The Dissertation Committee for Kanokkuan Chaicherdsakul
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Quantum Cosmological Correlations in Inflating Universe: Effect of Gravitational Fluctuation due to Fermion, Gauge, and Others Loops

Committee:

Steven Weinberg, Supervisor

Willy Fischler

Vadim Kaplunovsky

Sonia Paban

Paul Shapiro

Quantum Cosmological Correlations in Inflating Universe: Effect of Gravitational Fluctuation due to Fermion, Gauge, and Others Loops

by

Kanokkuan Chaicherdsakul, B.S.

Dissertation

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

Doctor of Philosophy

The University of Texas at Austin

December 2006

To ALL...

who have helped me

Acknowledgments

As I have read,

"I can see no scientific or logical reason not to seek consolation by adjustment of our beliefs–only the moral one, a point of honor. What do we think of someone who has managed to convince himself that he is bound to win a lottery because he desperately needs the money? Some might envy him his brief great expectations, but many others would think that he is failing in his proper role as an adult and rational human being, of looking at things as they are. In the same way that each of us has had to learn in growing up to resist the temptation of wishful thinking about ordinary things like lotteries, so our species has had to learn in growing up that we are not playing a starring role in any sort of grand cosmic drama."

S. Weinberg (from *Dreams of a final theory*)

I am grateful to

Prof. Steven Weinberg, my supervisor, for taking me as a thesis student when I was in the midst of a very difficult situation, for teaching me two cosmology classes and inflationary theories that most fascinate me, for supporting me on research and grant, for his recent papers and Weinberg theorems in Ref. [2, 3] that are the great helps of the extended calculation to Dirac, vector, and conformal scalar fields in this dissertation possible, for reading this dissertation, and for countless conversation on research subjects.

Prof. Bryce DeWitt, my previous supervisor, for helping me on my research qualifying exam, and for teaching me Heat Kernel technique and the In-In formalism until he became ill and passed away.

Jan Duffy, secretary of the theory group, for all her helpful administrative work and kind conversations.

Nantana and Theinchai Chaicherdsakul, my parents, for raising me up, for their encouragement and for deep love for my entire life.

Dr. Cheong Min Hong, my boyfriend, for correcting my English, for giving me rides to school and for preparing meals during my busy time.

Dr. S. N. Goenka, my scientific meditation teacher, for whom I learn what is important.

Sansua Gok Euang, San Euang Eia, Shipui Nia Nia, and KuanIn, my deepest gratitude.

Kanokkuan Chaicherdsakul

The University of Texas at Austin
December 2006

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Publication No. ____________________

Kanokkuan Chaicherdsakul, Ph.D.
The University of Texas at Austin, 2006

Supervisor: Steven Weinberg

Quantum theory of cosmological fluctuations with other matters is studied to higher order to understand the origin of the universe during the time of inflation. This study also links gravitational and all matter fluctuations with the observed cosmic microwave background (CMB) anisotropy. It is important to keep in mind that what is tested observationally is the paradigm that the primordial spectrum of inhomogeneities was nearly scale invariant and predominantly adiabatic. Therefore, if other matters such as fermion and gauge fields which do not drive inflation predict the scale invariant spectrums, their existence during inflation cannot be ruled out.

We therefore extend the calculation of quantum corrections to the cosmological correlation $\langle\zeta\zeta\rangle$, which has been done by Weinberg for a loop of minimally coupled scalars, to other types of matters loops and a general and realistic potential. This dissertation shows that departures from scale invariance are never large even when Dirac, vector, and conformal scalar fields are present *during* inflation and even when the two-loop spectrum is calculated. No fine tuning is needed, in the sense that effective masses can be arbitrary values. Although the loop power spectrum was generally expected to be smaller than the classical one by a factor of GH^2, I find that the quantum effect could be in the order of the classical value at the two loop level. The momentum dependence of the quantum spectrum goes as $q^{-3}\ln q$ for *all* massless matters at one-loop and goes as $q^{-3}\ln^2 q$ at two-loop. For massive matters, the momentum dependence goes as $q^{-3+\eta(m)}$, where $|\eta| \ll 1$ regardless of the value of m. Thus scale free correlations are consistent with natural reheating. These results imply that we and the things around us did not come from nothing or an unknown scalar field as in conventional beliefs. Rather it points to the fact that we originated from quantum fluctuations due to the interactions between gravity and various matters during the time of Big Bang inflation.

Contents

Chapter 1

Introduction

Where do we come from?

What are we?

Where are we going?

Paul Gauguin(1897)

1.1 What and Why?

If we like to understand where we and our surroundings such as electron and photon came from, it is natural to look back and ask what happened at the very early time during inflation. Gaussian and nearly scale invariant spectrum predicted by the scalar field dominated universe theory agreed very well with the current observation. It is therefore widely believed that the quantum fluctuation of scalar field during inflation seeded the large scale structure of the universe we observe today. The CMB primordial power spectrum is measured through the quantity $\langle\zeta\zeta\rangle$. This quantity ζ is important because of its relation to both matter and gravity and its prediction for the primordial density perturbation that is thought to be the origin of the structure of the universe. The observable quantity ζ_q is defined as

$$\zeta_q \equiv \frac{A_q}{2} - \frac{H\delta\rho_q}{\dot{\rho}} \tag{1.1}$$

to the linear order in the field equation of cosmological fluctuation[1]. This quantity is conserved outside horizon in inflation driven by a single scalar field. The theory of cosmological perturbation has been well known up to the quadratic term of action [1], which corresponds to the linear order in the field equations. Therefore, some authors in the literature may call it as linearized gravity. Recently the non-gaussian terms in the scalar field(s) have been calculated classically [7,8]. The quantum effect to arbitrary order in cosmological fluctuations has been more recently formulated by Weinberg[2]. With a sample massless minimal coupled scalar loop calculation, Weinberg's result shows that the momentum dependence goes as $q^{-3}\ln q$, with an additional suppression of G compared to the classical result. Is this true for other kinds of matters such as Dirac, vector, and conformal scalar fields as well? Also, what happens at higher-loop? In fact, the unbroken symmetry matters become non-negligible when we go beyond the quadratic term of action in the cosmological perturbation theories. It is therefore of great interest to investigate how the higher-order corrections to the bilinear correlation function $\langle\zeta\zeta\rangle_{loops}$ depend on momentum q when gravitational fluctuation interacts with general matters other than scalar field. Will the result go nearly as q^{-3} as in the scalar case?

We therefore study the possibilities of other matter that is not inflaton[2] during inflation. The minimal coupled scalar and gravitational fields are considered in

[1]See more details in [9]

[2]Although we do not actually know what inflaton is and is not, here we treat any field that has unbroken symmetries, i.e $\langle\chi\rangle = \langle\psi\rangle = \langle A_\mu\rangle = 0$ as other matter. A scalar field φ that has a non-zero expectation value is considered as an inflaton, as in conventional belief. Therefore in the quantum theory of cosmological fluctuation during inflation considered here, the inflaton φ, gravity $g_{\mu\nu}$, and other matters χ, ψ, A_μ are expanded as $\varphi = \bar{\varphi} + \delta\varphi, g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \chi = 0 + \delta\chi, \psi = 0 + \delta\psi, A_\mu = 0 + \delta A_\mu$, respectively.

many inflationary theories. However, in order to understand where other matters such as fermion and photon observed today came from, the inflaton would have to couple with these fields during inflation. There is no reason why there must be only scalar field and gravity but nothing else during inflation. Matter observed today would not have arisen if there was only a scalar field that coupled only to itself and gravity. If other matters such as Dirac, vector, and conformal scalar fields do not give anything far larger than the observed values[3] in the power spectrum, the existence of these fields during inflation may not be ruled out.

It is considered in the literatures that Dirac field cannot give a scale invariant primordial spectrum of density perturbation because its momentum dependence $\langle\zeta\zeta\rangle \sim q^{-1}$ is far from q^{-3}[6] and vector field can generate a scale invariant spectrum only if the mass square is negative and comparable to the Hubble scale during inflation[13]. However, the Dirac and vector fields in [6] and [13] respectively were considered *classically* only. In general, Dirac and vector fields in expanding universe only exist as quantum fields with zero expectation value $\langle\psi\rangle = \langle A_\mu\rangle = 0$ and what we observe is the density correlation function related to $\langle\zeta\zeta\rangle$, not the product of the fields $\langle\delta\bar{\psi}\delta\psi\rangle$, $\langle\delta A_i \delta A_j\rangle$. Therefore, we have to learn how to quantize such fields at higher order in cosmological fluctuation. The *quantum* effect to the observable $\langle\zeta\zeta\rangle$ due to Dirac and vector fields loops are calculated in this dissertation. The In-In formalism[2,4] appropriate for calculating expectation value rather than the S-Matrix in time dependent background is used. I mainly follow the calculation of Weinberg for the massless scalar field loop in [2] and extend it to (massive) Dirac, vector, and conformal scalar fields loops. In this dissertation, we investigate how the ζ correlation function depends on its momentum and whether it can be large. We can also find other theories in which the quantum effect can be as large as the classical

[3]The observed power spectrum today is $\langle\zeta\zeta\rangle = \frac{8\pi G H^2}{2(2\pi)^3 q^3 |\epsilon|}$

one, for instance by allowing the gravitational fluctuation to interact with additional massless scalar field with coupling in the order of M_{Pl}(i.e. $\mathcal{L}_{int} = M_{Pl}(\frac{\dot{\zeta}}{H} + 3\zeta)\sigma^3$). These interactions contribute two-loop sunset diagram in which the quantum correlation function does not get suppressed by an additional factor of GH^2.

It is also important to investigate the momentum dependence when quantum corrections are applied to the two point correlation functions. If the momentum dependence of the loop spectrum goes as q^{-n} such that n is far greater than 3, this will produce a larger spectrum than the classical value at outside horizon when $q \to 0$. The existence of Dirac and vector fields during inflation can be easily ruled out if the spectrum is far from scale invariant and therefore those fields cannot be the candidate for the origin of structure. We have shown that this is not the case. We *always* obtain nearly scale invariant spectrum even when Dirac, vector, and conformal scalar fields are included and even when the two-loop effect is calculated.

The organization of this dissertation is as follows. In chapter 2 we summarize the aspects of in-in formalism and the un-equal time (anti)commutators of all fluctuations that are needed for our present purposes. The general one-loop power spectrum formula is also derived. Chapter 3 introduces a class of theories, with a single inflaton field, plus additional massless and massive Dirac fields with gravitational and inflaton interactions and vanishing expectation values. Spin sum at different times for massive fermion is needed and explicitly calculated in that chapter. In addition, the late time behavior of Dirac field is considered through the solution of Dirac field equation in inflating universe and the momentum dependence of correlation function due to massless and massive fermion loop is calculated. In chapter 4 and 5 we follow the same approach as in chapter 3 except replacing the Dirac field with vector field and conformal scalar field respectively. Chapter 6 introduces a class of possible inflationary theories to illustrate the fact that one-loop quantum effect can be larger than what was previously thought but is still smaller than the observed values. In chapter 7 we calculate the two-loop correction of an observable correlation function $\langle\zeta\zeta\rangle_{two-loop}$. It is shown that for some theories such as when matter couples with both gravity and inflaton, the quantum effect can be as large as the classical value and gives a momentum dependence of $\frac{\ln^n q}{q^3}$ for general $n-$loop. The coefficient of $\frac{\ln^2 q}{q^3}$ is explicitly calculated at two-loop order. Chapter 8 summarizes all results we have in each chapters. The results show that for all theories and matters with a general potential $V(\varphi, \sigma, \bar{\psi}\psi, A_\mu A^\mu)$, the quantum correlation functions are never much larger than the classical (observed) value and are nearly scale invariant. In appendix A we derive the gravitational and general matters fluctuations used in one-loop calculation of chapter $3-6$ to the cubic order for the general reader. It is also explain there why the departure from scale invariance is still small even in a more general potential $V(\varphi) \to V(\varphi, \sigma, \bar{\psi}\psi, A_\mu A^\mu)$.

1.2 Simple Argument

There are some simple arguments that lead us to believe that quantum effect might contribute to the spectrum in the order of the observed values without getting suppressed by an additional factor of G. This could happen when matter couples with inflaton and gives vertex in the order of M_{Pl}. For example, consider the interaction of fermion ψ, inflaton φ, and gravity g via

$$\mathcal{L}_{int} = \sqrt{-g}\varphi\bar{\psi}\psi \tag{1.2}$$

In cosmological fluctuation, we normally expand the fields as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \varphi = \bar{\varphi} + \delta\varphi, \psi = 0 + \delta\psi \tag{1.3}$$

In general, fermion interacts with fluctuations of both gravity $\delta g_{\mu\nu}$ and scalar field $\delta\varphi$ and thus affects the conserved quantity ζ. However, we can choose a gauge which inflaton does not fluctuate ($\delta\varphi = 0$ gauge [7]) so that ζ is purely gravity in this gauge. Therefore, one of the interactions in eq. (1.2), for instance trilinear interaction, has the form

$$H_{\zeta\bar{\psi}\psi}(t) = \int d^3x a^3(t)\bar{\varphi}(t)\bar{\psi}(\mathbf{x},t)\psi(\mathbf{x},t)F[\zeta(\mathbf{x},t)] \tag{1.4}$$

where $F[\zeta]$ is some function of ζ, depending on the details after the expansion of the metric. Let us choose $F[\zeta] = \zeta$ for simplicity. Now we calculate the quantum contribution of $\langle\zeta\zeta\rangle$[4]

$$\Big\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle = -\int_{-\infty}^{t} dt_2 \int_{-\infty}^{t_2} dt_1 \Big\langle\Big[H_1, \Big[H_2, \zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big]\Big]\Big\rangle_0 \tag{1.5}$$

By solving Dirac field equation in inflating universe[5], the fermion pair $\bar{\psi}\psi$ goes as a^{-3} at late time. Therefore, the factor a^{-3} cancels with $\sqrt{-g}$ for each Hamiltonian. With zero factors of $a(t)$ the result of two time integrals grows as $(\ln a)^2$. But $(\ln a)^2$ is not as significant as $\bar{\varphi}$ in producing an appreciable effect in the interaction eq. (1.4). Since there are a total of four ζs on the RHS of eq. (1.5), we get the factor $|\zeta_q^o|^4 \simeq \frac{(8\pi G H^2)^2}{\epsilon^2 q^6}$. The two time integrals become

$$\int dt_1 \int dt_2 = \frac{1}{H^2}\int \frac{d\tau_2}{\tau_2}\int \frac{d\tau_1}{\tau_1} \simeq \frac{C}{H^2} \tag{1.6}$$

Since $\bar{\varphi}$ does not change very much during inflation, it can be approximated as $\bar{\varphi}(t_1) \simeq \bar{\varphi}(t_2) \simeq \bar{\varphi}(t_q)$ at the time of horizon exit so this does not enter into the time integral (in the same way as we approximate $H(t_1) \simeq H(t_2) \simeq H(t_q)$ during inflation). By collecting the factors of $H, \bar{\varphi}$ and $8\pi G$, we can then approximate the

[4] See the formula in chapter 2 or ref[2]
[5] This is shown explicitly in chapter 3

correlation function as

$$\langle\zeta\zeta\rangle_{loop} \to \frac{(8\pi G H^2(t_q))^2\bar{\varphi}^2(t_q)\mathcal{C}_q}{H^2\epsilon^2(t_q)} \tag{1.7}$$

where $\mathcal{C}_q$ is the momentum dependence that depends on the results of time and momentum integrals and the details of the propagator to the momentum p and p'. It is important to note that unlike the term H, the term $\bar{\varphi}$ which could arise via Yukawa coupling *does not* give spectrum suppressed by a factor of GH^2. To see this, let us take an example of the potential $V(\varphi) = \lambda_n\varphi^n$. Slow roll condition requires that

$$\frac{|V'|}{|V|} = \frac{n}{\bar{\varphi}} \ll \sqrt{8\pi G} \tag{1.8}$$

Alternatively we don't have to take this particular polynomial potential. In the Yukawa interaction considered above, $V(\varphi, \bar{\psi}\psi) = \varphi\bar{\psi}\psi \equiv \varphi f(\chi)$, where $f(\chi)$ is some scalar function if fermion pairs condensate. Slow roll condition also requires that

$$\frac{|V'|}{|V|} = \frac{f(\chi)}{\bar{\varphi}f(\chi)} = \frac{1}{\bar{\varphi}} \ll \sqrt{8\pi G} \tag{1.9}$$

Hence, $\bar{\varphi} \gg \frac{1}{\sqrt{8\pi G}}$ in both cases. So we have the correlation function

$$\langle\zeta\zeta\rangle_{loop} \to \frac{(8\pi G)H^2\mathcal{C}_q}{\epsilon^2(t_q)} \tag{1.10}$$

which is in the order of classical result.

With large vertices in the order of the M_{Pl}, it raises the possibility that quantum effect is *not* suppressed by a factor of G as was previously believed and therefore contribute to the loop spectrum in the order of classical value. However, such realistic $\bar{\varphi}(t_q) \sim M_{Pl}$ coupling can only happen in massive, but not massless, matter fields at one-loop level. The reason is that inflaton fluctuates around non-zero background that always contributes to the non-derivative matter terms to the second order after field expansions, i.e., $m\bar{\psi}\psi = \bar{\varphi}\bar{\psi}\psi$ or $|\bar{\varphi}|^2 A_i^2 = m^2 A_i^2$. The massive case is more general because it allows the possibilities of interactions with other broken symmetry fields such as inflaton and hence gives large vertices in the order of M_{Pl}. Although the argument above is valid, we still need to find out what $\mathcal{C}_{\rm II}$ is through actual calculations because $\mathcal{C}_{\rm II}$ is also a function of mass that arises through the massive propagators. We have to investigate whether this $\mathcal{C}_{\rm II}(m)$ gives other kind of suppression.

Chapter 2

Loop In-In Correlation Function

For in the case of beings like ourselves,
death is certain,
life is uncertain;
All existing things are transitory
and subject to decay.
Therefore be heedful of your ways day and night. Khema

Details of in-in formalism are well explained in refs.[2] and [4]. For the purpose of calculation in this dissertation, we will briefly review some important aspects of the in-in formalism and extend a one-loop formulae to general matters such as fermion and gauge fields in this chapter.

2.1 In-In Formalism in Cosmology and Weinberg's Formula

In any quantized theory that involves gravity such as inflationary theories, there is a need of in-in formalism for the following reasons

1. The in-states (the remote past) are not the same as the out-states (the remote future) in general. The standard way of quantization normally assumes the space-time to be asymptotically flat both in the remote past and remote future. However, among many problems that involve gravity such as inflationary theories or black hole singularities, there are only rare cases which the final or initial states are fully known. So our approach is to calculate an observable such as the correlation function with a fixed initial(final) state and then see what the result is at time t, rather than assuming something that is not certainly known in a final (initial) state. In inflationary theories, this means we fix an initial condition, when the wavelength is deep inside horizon, and calculate an observable as the time evolves. We do not assume anything about the final state, in particular when the wavelength is outside horizon.

2. The in-out correlation functions and quantum effective action are not real and causal in general, especially in the theories that involve gravity. The in-in results are always real and causal such that the future can only arise from the past, not from the future.

3. A fluctuation Hamiltonian governing the time-dependence of field fluctuations is explicitly time dependent because of the rapid expansion of the universe.

An observable in expanding universe we should calculate is the expectation value, instead of S-Matrix. The in-in expectation values of some product $Q(t)$ of field operators, all at the same time t is[2]

$$\left\langle Q(t) \right\rangle = \left\langle \bar{T} e^{i \int_{-\infty}^{t} H_I(t')dt'} Q^I(t) T e^{-i \int_{-\infty}^{t} H_I(t')dt'} \right\rangle_0 \tag{2.1}$$

We can expand the equation above as a sum of products of bilinear products with the use of Wick theorem. This leads to new in-in Feynman rules that are different from the conventional in-out Feynemn rules. The main differences are:

1. Time Dependent Vertices: They are distinguished between the "right" (called R) and "left" (called L) vertices, arising from the time and anti-time order products respectively. The R and L vertices contribute a factor of $iV(t)$ and $-iV(t)$ respectively.

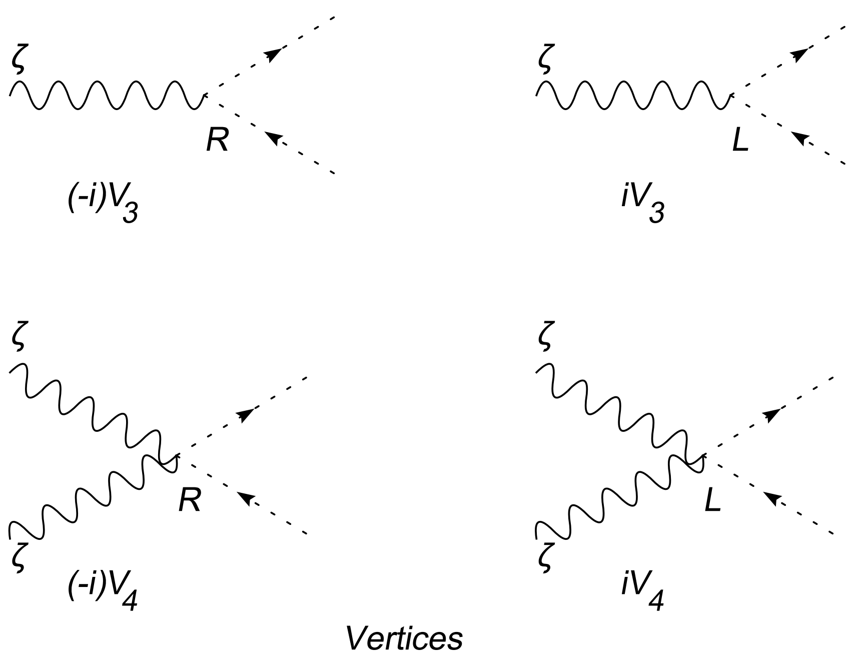

Figure 2.1: There are two vertices in each interactions(R and L vertices). Extra minus sign is needed for vertex L

2. Unequal-Time Propagators: There are $-i\Delta^{RR} = \langle TA(\mathbf{x},t)B(\mathbf{x}',t')\rangle$, $i\Delta^{LL} =$

$\langle \bar{T} A(\mathbf{x},t) B(\mathbf{x}',t') \rangle$, $i\Delta^{RL} = \langle A(\mathbf{x},t) B(\mathbf{x}',t') \rangle$, and $i\Delta^{LR} = \langle B(\mathbf{x},t) A(\mathbf{x}',t') \rangle$ propagators, arising from pairing the field operators in eq. (2.1). Note that there is an extra minus sign when switching the anti-commuting field operators.

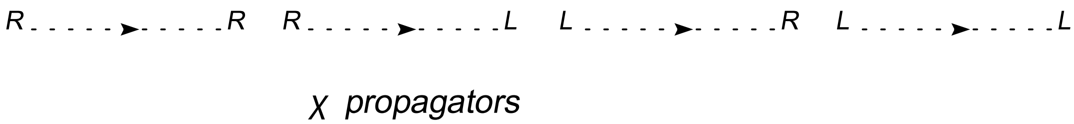

Figure 2.2: There are a total of four propagators for $\Delta^{RR}, \Delta^{RL}, \Delta^{LR}$ and Δ^{LL}

A diagram with N vertices contributes 2^N ways of choosing each vertex to be a left vertex or a right vertex. Therefore, the numbers of in-in diagrams are more than the numbers of conventional in-out diagrams for each topology. However, we do not have to calculate all 2^N diagrams in the correlation functions. For example, for 2 vertices, there are only two diagrams RR and RL to calculate instead of all diagrams RR, RL, LR and LL. The reasons are that

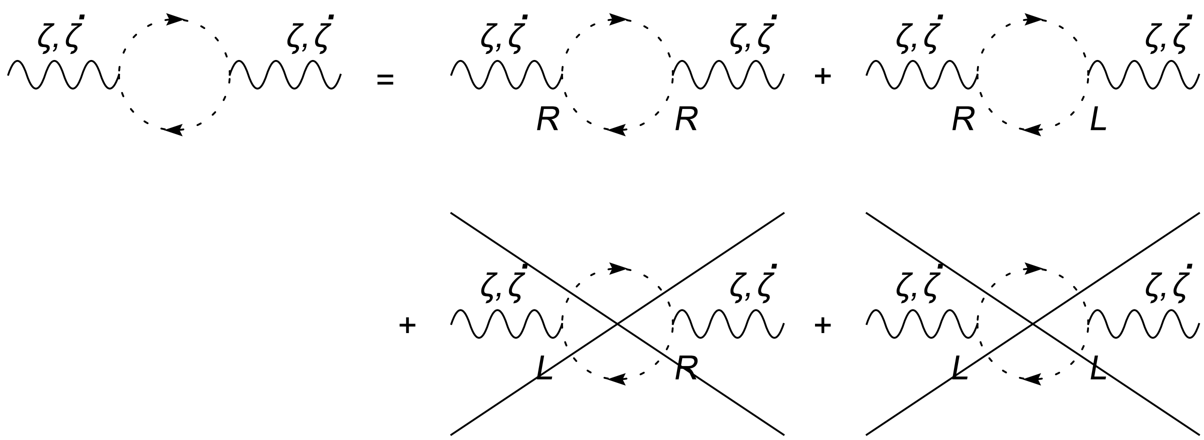

Figure 2.3: Only RR and RL need to be calculated

1) Although the diagrams of Δ^{LL} connected to $\zeta_L \zeta_L$ without ζ_R contribute to the quantum effective action, they do not contribute to the quantum correlation func-

tions and quantum effective field equations. Even though the formalism was constructed to list all the field operators in R and L in the (effective) action, the only thing that gives physical information is the correlation function and effective equation arising from varying the effective action. In path integral approach, all the "L" fields are set equal to R fields (which we then further define as the physical fields) after varying the in-in quantum effective action with respect to R fields.[1] Therefore, any diagram in the effective action which have only L fields but no R field does not contribute to the effective equation or correlation function.

2) The propagators Δ^{RL} and Δ^{LR} are related by some symmetry and its conjugate. One therefore does not need to calculate loops of LR again once the loops of RL are known.

The diagram method mentioned above is for those who prefer path-integral quantization. For canonical quantization, we can expand eq. (2.1) and many terms are cancelled due to the existence of anti-time order factor in eq. (2.1). This leads us to Weinberg's formula[2]

$$\Big\langle Q(t)\Big\rangle = \sum_{n=0}^{\infty} i^N \int_{-\infty}^{t} dt_N \int_{-\infty}^{t_N} dt_{N-1}... \int_{-\infty}^{t_2} dt_1 \Big\langle \Big[H_I(t_1), \Big[H_I(t_2), ... \Big[H_I(t_N), Q^I(t)]...\Big]\Big]\Big\rangle_0 \tag{2.2}$$

where the expectation value in the RHS of equation above is in free field vacuum annihilated by annihilation operators and that in the LHS is in interacting vacuum. For the rest of this dissertation, we will use eq. (2.2) above to calculate loop correlation function.

2.2 Unequal Time (Anti) Commutators of all Matters

To quantize in time dependent background, we need the unequal time (anti) commutators.

[1] This gives the same result as varying the action with respect to "L" fields and setting "R" fields equal to "L" fields.

For a real scalar field $\sigma(\mathbf{x},t) = \sigma^*(\mathbf{x},t)$, we have

$$\Big[\sigma(\mathbf{x}_1,t_1),\sigma(\mathbf{x}_2,t_2)\Big] = \int d^3p e^{i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)}\Big(\sigma_{\mathbf{p}}(t_1)\sigma^*_{\mathbf{p}}(t_2) - \sigma^*_{-\mathbf{p}}(t_1)\sigma_{-\mathbf{p}}(t_2)\Big) \quad (2.3)$$

For a complex scalar field $\chi(\mathbf{x},t) \neq \chi^*(\mathbf{x},t)$, the commutator is also the same as that of real scalar field.

$$\Big[\chi(\mathbf{x}_1,t_1),\chi^*(\mathbf{x}_2,t_2)\Big] = \int d^3p e^{i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)}\Big(\chi_{\mathbf{p}}(t_1)\chi^*_{\mathbf{p}}(t_2) - \chi^*_{-\mathbf{p}}(t_1)\chi_{-\mathbf{p}}(t_2)\Big) \quad (2.4)$$

but the existence of anti-particle also requires that

$$\begin{aligned}\Big[\chi^*(\mathbf{x}_1,t_1),\chi(\mathbf{x}_2,t_2)\Big] &= \int d^3p e^{i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)}\Big(\chi_{\mathbf{p}}(t_1)\chi^*_{\mathbf{p}}(t_2) - \chi^*_{-\mathbf{p}}(t_1)\chi_{-\mathbf{p}}(t_2)\Big) && (2.5)\\ &= \Big[\chi(\mathbf{x}_1,t_1),\chi^*(\mathbf{x}_2,t_2)\Big] && (2.6)\end{aligned}$$

For fermion $\psi(\mathbf{x},t)$ and its conjugate $\bar{\psi}$

$$\Big\{\psi(\mathbf{x}_1,t_1),\bar{\psi}(\mathbf{x}_2,t_2)\Big\} = \int d^3p e^{i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)}\sum_s\Big(U_{\mathbf{p},s}(t_1)\bar{U}_{\mathbf{p},s}(t_2) + V_{-\mathbf{p},s}(t_1)\bar{V}_{-\mathbf{p},s}(t_2)\Big) \quad (2.7)$$

The difference from bosonic field is that fermion satisfies anti-commutator rule. So, the dominant and dominant mode solutions do not cancel at late time as in the bosonic case. Unlike the case of charged scalar, fermion requires the existence of anti-particle but the anti-commutator is not equal to its conjugate,

$$\begin{aligned}\Big\{\bar{\psi}(\mathbf{x}_1,t_1),\psi(\mathbf{x}_2,t_2)\Big\} &= \int d^3p e^{i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)}\sum_s\Big(\bar{V}_{\mathbf{p},s}(t_1)V_{\mathbf{p},s}(t_2) + \bar{U}_{-\mathbf{p},s}(t_1)U_{-\mathbf{p},s}(t_2)\Big) && (2.8)\\ &\neq \Big\{\psi(\mathbf{x}_1,t_1),\bar{\psi}(\mathbf{x}_2,t_2)\Big\} && (2.9)\end{aligned}$$

For a real vector field $A_i(\mathbf{x},t)$,

$$\Big[A_i(\mathbf{x}_1,t_1),A_j(\mathbf{x}_2,t_2)\Big]=\int d^3pe^{i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)}\Pi_{ij}\Big(\mathcal{A}_{\mathbf{p}}(t_1)\mathcal{A}^*_{\mathbf{p}}(t_2)-\mathcal{A}^*_{-\mathbf{p}}(t_1)\mathcal{A}_{-\mathbf{p}}(t_2)\Big) \tag{2.10}$$

where Π_{ij} is the polarization factor in which

$$\Pi_{ij}(\hat{p})=\sum_{\lambda=1}^{2}\hat{e}^*_i(\hat{p},\lambda)\hat{e}_j(\hat{p},\lambda)=\delta_{ij}-\frac{p_ip_j}{|\mathbf{p}|^2} \tag{2.11}$$

which is time independent for $m=0$

2.3 Normal Ordering

In this section, we will normal order the product of $2N$ field operators (with $N=2$ as an example) used in the product of Hamiltonian $H_1...H_N$ in eq. (2.2). We need to normal order the product of field operators but there is no need to time order them as in the standard in-out theory. The is because eq. (2.1) also has a factor of anti-time ordering. Some cancellations would occur, leaving us with the simpler formula eq. (2.2) in which we no longer need time ordering.

The interaction Hamiltonian always has at least two matter fields and one gravitational fluctuation ζ. In the next two subsections we will consider the product of various type matter field operators.

2.3.1 Real Field

We define the field operator $R_i\equiv R(\mathbf{x}_i,t_i)$ as a linear combination of positive frequency and negative frequency

$$R=R^++R^- \tag{2.12}$$

For each interaction Hamiltonian,

$$H_i = R_i R_i F[\zeta] \tag{2.13}$$

Hence, the product of $2N$ field operator is

$$R_1 ... R_{2N} \quad = \quad : R_1 ... R_{2N} : + \sum_{perm} [R_i^+, R_j^-][R_k^+, R_l^-] \tag{2.14}$$

For example, the product of the four field operators for $N = 2$ is

$$R_1 R_2 R_3 R_4 \quad = \quad : R_1 R_2 R_3 R_4 : + [R_1^+, R_2^-][R_3^+, R_4^-] \tag{2.15}$$

$$+ [R_1^+, R_3^-][R_2^+, R_4^-] + [R_2^+, R_3^-][R_1^+, R_4^-] \tag{2.16}$$

where $R^\pm$ are the field operators associated with annihilation and creation operators, respectively, such that

$$R^+|0\rangle = \langle 0|R^- = 0 \tag{2.17}$$

The actual calculation will be less complicated than the equations presented above. The reason is that in each interaction Hamiltonian H_i, there will be at least two products of field in the same space and time $\mathbf{x}_i$ and t_i as seen from eq. (2.13) but at different momentum. Therefore eq. (2.15) is reduced to

$$R_1 R_1 R_2 R_2 \quad = \quad : R_1 R_1 R_2 R_2 : + [R_1^+, R_1^-][R_2^+, R_2^-] \tag{2.18}$$

$$+ 2[R_1^+, R_2^-][R_1^+, R_2^-] \tag{2.19}$$

The non-trivial commutators are the last term which is at different space and time. Hence,

$$\langle R_1 R_1 R_2 R_2 \rangle_0 \quad = \quad 2[R_1^+, R_2^-][R_1^+, R_2^-] \tag{2.20}$$

since the vacuum expectation value of the normal product of field operator is always zero.

2.3.2 Complex Field

For the charged bosonic or fermion field operator $C(\mathbf{x}_i, t_i) \neq C^\dagger(\mathbf{x}_i, t_i)$, we have

$$C_1^\dagger C_2 C_3^\dagger C_4 \quad = \quad :C_1^\dagger C_2 C_3^\dagger C_4: + [C_1^{-\dagger}, C_2^-]_\mp [C_3^{-\dagger}, C_4^-]_\mp \tag{2.21}$$

$$+ [C_2^+, C_3^{+\dagger}]_\mp [C_1^{-\dagger}, C_4^-]_\mp \tag{2.22}$$

where the $\mp$ sign means the commutator and anti-commutator for bosonic and fermionic fields, respectively. The C^+ and C^- are the terms in C that destroy particles and create anti-particle respectively, such that

$$C^+|0\rangle = C^{-\dagger}|0\rangle = \langle 0|C^- = \langle 0|C^{+\dagger} = 0 \tag{2.23}$$

For each Hamiltonian interaction, H_i must have at least $C_i^\dagger C_i$ in equal space and time. Therefore, when $1 = 2$, $3 = 4$, the non-trivial (anti)commutators becomes

$$\langle C_1^\dagger C_1 C_2^\dagger C_2 \rangle_0 \quad = \quad [C_1^+, C_2^{+\dagger}]_\mp [C_1^{-\dagger}, C_2^-]_\mp \tag{2.24}$$

As seen from the equation above, unlike the standard in-out theory there is *no* " $-$ " sign when we close the fermion loop. The in-in formula in eq. (2.2) has implied the cancellation of some certain diagrams. The combination of all diagrams with the same topology are already contained in eq. (2.2). For example, for the diagrams which has two external legs and trilinear vertices, the combination of the two diagrams with the same topology (one is right-right vertices and another is left-right vertices) gives us an expression in eq. (2.49) in the next section.

2.4 General Matter Loop Power Spectrum Formula

In this section we calculate a loop power spectrum of general matter valid for scalar, fermion and gauge fields. We will use this loop power spectrum in the later chapters. To see this in more detail, let us consider a general complex bosonic or fermionic field ψ with the interaction[2]

$$H_I(t) = \int d^3x V(t)\psi^*(\mathbf{x},t)\psi(\mathbf{x},t)\zeta(\mathbf{x},t) \tag{2.25}$$

where we omit the spinor and vector indices in this section and $V(t)$ is any time dependent vertex function as a consequence of expanding universe. For instance $V(t) = a^3(t)\bar{\varphi}(t)$. The second order in Hamiltonian of eq. (2.2) is

$$\Big\langle Q(t)\Big\rangle = -\int_{-\infty}^{t} dt_2 \int_{-\infty}^{t_2} dt_1 \Big\langle \Big[H_1, \Big[H_2, Q\Big]\Big]\Big\rangle_0 \tag{2.26}$$

If Q is the product of the gravitational field ζ treated as the external legs, the vacuum expectation values of the matter, either boson or fermion, circulated inside the loop in the RHS of eq. (2.26) can be evaluated independently from Q because they have different types of creation and annihilation operators. For $Q = \zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)$, we can write each field in interaction picture in terms of creation and annihilation operators as

$$\zeta(\mathbf{x},t) = \int d^3p(e^{i\mathbf{p}\cdot\mathbf{x}}\alpha_{\mathbf{p}}\zeta_p(t) + e^{-i\mathbf{p}\cdot\mathbf{x}}\alpha^*_{\mathbf{p}}\zeta^*_p(t)) \tag{2.27}$$

$$\psi(\mathbf{x},t) = \int d^3p \sum_\lambda (e^{i\mathbf{p}\cdot\mathbf{x}}\alpha_{\mathbf{p},\lambda}U_{\mathbf{p},\lambda}(t) + e^{-i\mathbf{p}\cdot\mathbf{x}}\beta^\dagger_{\mathbf{p},\lambda}V_{\mathbf{p},\lambda}(t)) \tag{2.28}$$

and

$$\psi^*(\mathbf{x},t) = \int d^3p \sum_\lambda (e^{-i\mathbf{p}\cdot\mathbf{x}}\alpha^\dagger_{\mathbf{p},\lambda}U^*_{\mathbf{p},\lambda}(t) + e^{i\mathbf{p}\cdot\mathbf{x}}\beta_{\mathbf{p},r}V^*_{\mathbf{p},\lambda}(t)) \tag{2.29}$$

[2]We can easily generalize to other type of interactions (such as the terms containing field derivative) later in the next chapter, once we obtain a general formula valid for any matter in this section.

where $\alpha_{\mathbf{p}}$ satisfies the commutation relations since ζ is boson but $\alpha_{\mathbf{p},\lambda}$ and $\beta_{\mathbf{p},\lambda}$ satisfy the (anti) commutation relations for (fermion) boson matter loop. λ is either the spin index for the fermion or the helicity for the gauge field.

Let us now evaluate the RHS of eq. (2.26)

$$\Big[H_2, Q\Big] = \int d^3x_2 V_2 \Big[\psi_2^* \psi_2 \zeta_2, Q\Big] = \int d^3x_2 V_2 \psi_2^* \psi_2 (\zeta_2 Q - Q\zeta_2) \tag{2.30}$$

$$\begin{aligned}
\Big\langle \Big[H_1, \Big[H_2, Q\Big]\Big]\Big\rangle_0 &= \int\int d^3x_1 d^3x_2 V_1 V_2 \Big\langle \Big[\psi_1^*\psi_1\zeta_1, \psi_2^*\psi_2\zeta_2 Q\Big] && (2.31)\\
&\quad -\Big[\psi_1^*\psi_1\zeta_1, Q\psi_2^*\psi_2\zeta_2\Big]\Big\rangle_0 && (2.32)\\
&= \int\int d^3x_1 d^3x_2 V_1 V_2 \Big\langle \Big(\psi_1^*\psi_1\psi_2^*\psi_2(\zeta_1\zeta_2 Q - \zeta_1 Q\zeta_2) && (2.33)\\
&\quad +\psi_2^*\psi_2\psi_1^*\psi_1(Q\zeta_2\zeta_1 - \zeta_2 Q\zeta_1)\Big)\Big\rangle_0 && (2.34)
\end{aligned}$$

Since ψ is independent from ζ and Q, the vacuum expectation value can be evaluated independently.

For the ζ part,

$$\langle \zeta_1\zeta_2 Q\rangle_0 = 2\int d^3k d^3k' e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x})+i\mathbf{k}'\cdot(\mathbf{x}_2-\mathbf{x}')} \zeta_k(t_1)\zeta_{k'}(t_2)\zeta_k^*(t)\zeta_{k'}^*(t) \tag{2.35}$$

Hence,

$$\langle \zeta_1\zeta_2 Q\rangle_0 = \langle Q\zeta_2\zeta_1\rangle_0^* \tag{2.36}$$

Similarly,

$$\langle \zeta_1 Q\zeta_2\rangle_0 = 2\int d^3k d^3k' e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x})+i\mathbf{k}'\cdot(\mathbf{x}_2-\mathbf{x}')} \zeta_k(t_1)\zeta_{-k'}(t)\zeta_k^*(t)\zeta_{-k'}^*(t_2) \tag{2.37}$$

Hence,

$$\langle \zeta_1 Q\zeta_2\rangle_0 = \langle \zeta_2 Q\zeta_1\rangle_0^* \tag{2.38}$$

For matter field, only ψ_1 could pair with ψ_2 at different time but at the same momentum (this corresponds to the propagator at different time), not ψ_1 with ψ_1^* because they are at different momentum. To see this in more details, we write the field operator ψ in terms of creation and annihilation operators similar to that in eq. (2.28). Hence, these products of the fields can be written in momentum space as

$$\langle\psi_1^*\psi_1\psi_2^*\psi_2\rangle_0 = \int\int d^3p d^3p' e^{i(\mathbf{p}+\mathbf{p}')\cdot(\mathbf{x}_1-\mathbf{x}_2)} \sum_{\lambda,\lambda'} U_{\mathbf{p},\lambda}(t_1)U^*_{\mathbf{p},\lambda}(t_2)V_{\mathbf{p}',\lambda'}(t_2)V^*_{\mathbf{p}',\lambda'}(t_1) \tag{2.39}$$

Since

$$\langle\psi_1^*\psi_1\psi_2^*\psi_2\rangle_0 = \langle\psi_2^*\psi_2\psi_1^*\psi_1\rangle_0^* \tag{2.40}$$

Hence, the correlation function is

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}\Big\langle Q(t)\Big\rangle = -\int_{-\infty}^{t} dt_2 \int_{-\infty}^{t_2} dt_1 \int d^3x \int d^3x_2 \int d^3x_1 \tag{2.41}$$

$$\times 2Re\Big(\langle\psi_1^*\psi_1\psi_2^*\psi_2\rangle_0(\langle\zeta_1\zeta_2 Q\rangle_0 - \langle\zeta_1 Q\zeta_2\rangle_0)\Big) \tag{2.42}$$

Note that many integrals over momentum $\mathbf{k}$ and $\mathbf{k}'$ can be eliminated via the space integrations that produce delta functions, i.e.,

$$\int d^3x \quad\to\quad (2\pi)^3\delta^3(\mathbf{q}-\mathbf{k}) \tag{2.43}$$

$$\int d^3x_1 \quad\to\quad (2\pi)^3\delta^3(\mathbf{q}+\mathbf{p}+\mathbf{p}') \tag{2.44}$$

$$\int d^3x_2 \quad\to\quad (2\pi)^3\delta^3(\mathbf{k}'-\mathbf{p}-\mathbf{p}') \tag{2.45}$$

Since $\mathbf{q} = -\mathbf{k}'$, $\mathbf{x}'$ in the exponential of the integrand is cancelled. We therefore have the formula

$$\int d^3xe^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}\Big\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle_{loop} = -4(2\pi)^9\int d^3pd^3p'\delta^3(\mathbf{q}+\mathbf{p}+\mathbf{p}') \tag{2.46}$$

$$\int_{-\infty}^{t} dt_2V_2\int_{-\infty}^{t_2} dt_1V_1Re\Big(\mathcal{Z}\mathcal{M}\Big) \tag{2.47}$$

where

$$\mathcal{Z} = \zeta_q(t_1)\zeta_q^*(t)\Big(\zeta_q(t_2)\zeta_q^*(t) - \zeta_q(t)\zeta_q^*(t_2)\Big) \tag{2.48}$$

and

$$\mathcal{M} = \sum_{\lambda,\lambda'} U_{p,\lambda}(t_1)U^*_{p,\lambda}(t_2)V_{p',\lambda'}(t_2)V^*_{p',\lambda'}(t_1) \tag{2.49}$$

Eq. (2.49) is a formula for general matter field loop.

For a real scalar field σ, we have $U_p = \sigma_p$ and $V_p = \sigma_p^*$. Hence, eq. (2.49) becomes

$$\mathcal{M}_\sigma = 2\sigma_p(t_1)\sigma_p^*(t_2)\sigma^*_{p'}(t_2)\sigma_{p'}(t_1) \tag{2.50}$$

Note that we have an additional factor of 2 for any real field, as seen from eq. (2.20). For charged scalar field $\chi(\mathbf{x},t) \neq \chi^*(\mathbf{x},t)$, we still have $U_p = \chi_p$ and $V_p = \chi_p^*$. Hence, eq. (2.49) becomes

$$\mathcal{M}_\chi = \chi_p(t_1)\chi_p^*(t_2)\chi^*_{p'}(t_2)\chi_{p'}(t_1) \tag{2.51}$$

which only differs from the real scalar field by a factor of 2.

For fermionic field which is always complex, eq. (2.49) becomes

$$\mathcal{M}_\psi = \sum_{r,s=1,2} U_{p,r}(t_1)U^*_{p,r}(t_2)V_{p',s}(t_2)V^*_{p',s}(t_1) \tag{2.52}$$

For real vector field, we have $U_{p,\lambda} = \mathcal{A}_{p,\lambda}$ and $V_{p,\lambda} = \mathcal{A}^*_{p,\lambda}$. Hence, eq.(2.49) becomes

$$\mathcal{M}_\mathcal{A} = 2\sum_{\lambda,\lambda'} \mathcal{A}_{p,\lambda}(t_1)\mathcal{A}^*_{p,\lambda}(t_2)\mathcal{A}^*_{p',\lambda'}(t_2)\mathcal{A}_{p',\lambda'}(t_1) \tag{2.53}$$

where the helicity index $\lambda = 1, 2$ for massless vector field and $\lambda = 1, 2, 3$ for massive vector field.

Note that the fermion and gauge fields require the summation over spin and helicity at *different time*, especially in the massive theories. We will take this requirement into account and use the formula for general matter field loop shown above to calculate the power spectrums for more realistic interactions between various kinds of matter and gravity in subsequent chapters.

2.5 Some Technical Difficulties

Although the formula for general matter field loop seems to be a convenient and compact way to calculate loop power spectrum, we still need to integrate over times and momentums to determine the momentum dependence. The integration for the case of massless matter loop poses no difficulty since the propagator is a simple enough function to solve exactly. Challenges arise when the matter has non-zero mass because the solution of field equations consists of Hankel functions with various orders ν of various types of matter. There is product of two Hankel functions at different times t_i, t_j, in which $i \neq j$, for each propagator line of momentum p. This means that it requires product of *at least* four Hankel functions [3]to be integrated over times and momentums. Although it appears that we only need to integrate over product of two Hankel functions at a time (for example, when we first integrate over time t_1), the result of the first time integral is in general the product of two Hankel functions with the upper limit t_2[4]. Therefore, we now need to integrate over product of four Hankel functions in the next time integral, in which, two come from the original function at time t_2 and the other two come from the result of the time integral t_1 with upper limit t_2. This adds up to product of $2n$ Hankel functions that we must integrate in the last t_n integral. The results of those time integrals, if available, becomes the integrands to integrate over various spatial momentum ps as well. One would also need to regulate UV divergence with such Bessel's function propagators at various orders for various matters. For massive fermion and gauge fields, we also have to sum over spins and helicities at different times because the propagators at each momentum are at different times. The entire process quickly becomes tedious and difficult for us to determine the momentum dependence of loop power spectrum precisely.

The difficulties may be reduced if we only integrate the momentum mode ps up to some cut off momentum Λq. This allows us to approximate the various Hankel functions to be some simple enough functions for integration. This means the momentum modes of matter ps circulated inside the loop exit the horizon before or around the same time as the momentum mode q of ζ. As mentioned in [2], it is only if the integrals over internal wave number p are dominated by values of the order $p \approx q$ that we can speak of a definite time of horizon exit, where $\frac{q}{a} \approx \frac{p}{a} \approx H$. For the massive theories considered here, such approximation of Hankel functions is necessary to allow us to integrate and determine the momentum dependence of

[3]Four Hankel functions for the two point functions considered here. For n point functions, we have to integrate over product of $2n$ Hankel functions inside the loops.

[4]There is no problem with the lower limit of time integral when $t \to -\infty$ since it exponentially decays away after rotating to imaginary axis.

loop spectrum.

Chapter 3

Dirac Field

Pleasure and pain keep coming, like day and night.

Why then cast away, your peace of mind, oh child?

unwanted things may come our way, wanted things as well.

Life contains both light and shade. Then why dance? Why weep?

In life they keep coming, autumn and springs.

If the mind does not waver, you enjoy infinite happiness. S.N.Goenka

3.1 Fermion, Inflaton, and Gravity

In the known theory of cosmological fluctuation, only inflaton and gravity are considered. The fermion, inflaton and Einstein gravity actions considered here are

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_g + \mathcal{L}_\varphi + \mathcal{L}_f && (3.1)\\
&= -\frac{1}{2}\sqrt{-g}[\frac{1}{8\pi G}R + g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + 2V && (3.2)\\
&\quad +\bar{\psi}\gamma^\alpha(\mathcal{D}_\alpha\psi) - (\mathcal{D}_\alpha\bar{\psi})\gamma^\alpha\psi + 2m\bar{\psi}\psi] && (3.3)
\end{aligned}$$

where we add a total derivative term $\mathcal{D}_\alpha(\sqrt{-g}\bar{\psi}\gamma^\alpha\psi)$ to get the symmetry of fermion energy momentum tensor in which we later need. $\mu, \nu, ...$ are the space time indices, and $\alpha, \beta, ...$ are the Lorentz indices raised and lowered by the vierbein V^α_μ. To deal with fermion, we need Tetrad formalism[10]. The metric in any general non-inertial coordinate system is related to vierbein by

$$g_{\mu\nu}(x) = V^\alpha_\mu(x) V^\beta_\nu(x) \eta_{\alpha\beta} \tag{3.4}$$

The covariant derivative to fermionic field due to the gravity interaction is

$$\mathcal{D}_\alpha \equiv V^\mu_\alpha \partial_\mu + \frac{1}{2}\sigma^{\beta\gamma} V^\nu_\beta V^\mu_\alpha V_{\gamma\nu;\mu} \tag{3.5}$$

In ADM formalism [5] where we slice the space and time to $3+1$, the general metric is given by

$$g_{\mu\nu} = \begin{pmatrix} N_i N^i - N^2 & N_j \\ N_i & g_{ij} \end{pmatrix} \tag{3.6}$$

The inverse metric is

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^j}{N^2} \\ \frac{N^i}{N^2} & g^{ij} - \frac{N^i N^j}{N^2} \end{pmatrix} \tag{3.7}$$

where

$$N_i = g_{ij} N^j \tag{3.8}$$

We can choose to fix the gauge at the matter side $\delta\varphi = B = 0$ [7], instead of at the gravity side. With a gauge that inflaton does not fluctuate, the observable quantity ζ is purely gravity and hence fermionic fluctuation implicitly affects this observable through the gravitational fluctuation in the action. The gravitational field and inflaton actions can be written in ADM form as

$$\mathcal{L}_g + \mathcal{L}_\varphi = \frac{a^3 e^{3\zeta}}{2}[NR^{(3)} - 2NV + N^{-1}(K^i_j K^j_i - (K^i_i)^2) + N^{-1}\dot{\bar{\varphi}}^2] \tag{3.9}$$

where

$$K_{ij} \equiv \frac{1}{2}[\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i] \tag{3.10}$$

3.2 Constraint and Field Equations

Since the time derivatives of N and N^i never occur, they plays the role of Lagrange multipliers. Varying the actions with respect to N and N^i give constraint equations. By solving the constraint equations or the Einstein equations (if they are written in terms of A, E, F) to the first order in fields[1], the auxiliary fields N and N^i become

$$N = 1 + \dot{\zeta}/H \tag{3.11}$$

and

$$N^i = -\frac{1}{a^2 H}\partial_i \zeta + \epsilon \partial_i \nabla^{-2} \dot{\zeta} \tag{3.12}$$

Note that, in general, the metrics N and N^i should be functions of ψ and $\bar{\psi}$. However, for the purpose of getting trilinear vertices or action to the cubic order, we only need gravitational fluctuation to the linear order since the zero expectation value matter such as fermion ($\langle\psi\rangle = 0$) is already considered as second order in the action and energy momentum tensor. We get the same result of N and N_i by solving linearized Einstein's equation, which is the same as the case of matter fields with no unbroken symmetry.

[1]See appendix for more details

The fermionic action can be varied with respect to ψ and $\bar{\psi}$ independently. We have

$$(\gamma^{\alpha}\mathcal{D}_{\alpha}+m)\psi=0 \tag{3.13}$$

In general ADM metric, we can solve eqs. (3.4) and (3.6) and choose the vierbein to be

$$V_k^0=0, V_0^0=N, V_0^m=aN^m e^{\zeta}, V_i^m=ae^{\zeta}\delta_i^m \tag{3.14}$$

$$V_m^0=0, V_0^0=N^{-1}, V_0^k=-\frac{N^k}{N}, V_m^i=a^{-1}e^{-\zeta}\delta_m^i \tag{3.15}$$

$$V_{0i}=0, V_{00}=-N, V_{m0}=a^{-1}N^m e^{-\zeta}, V_{mi}=ae^{\zeta}\delta_{mi} \tag{3.16}$$

For free field in the interaction picture, we can use the vierbein above with $N=1, N^i=0$. Therefore, the fermionic field equation in expanding universe is

$$\gamma^0\dot{\psi}+\frac{3H\gamma^0}{2}\psi+\frac{\gamma^i\partial_i}{a}\psi+m\psi=0 \tag{3.17}$$

or

$$a^{-\frac{3}{2}}\frac{d}{dt}(a^{\frac{3}{2}}\gamma^0\psi)+\frac{\gamma^i\partial_i}{a}\psi+m\psi=0 \tag{3.18}$$

To solve Dirac equation in expanding universe, we can re-scale the field $\psi\equiv a^{-\frac{3}{2}}\chi$ and work with conformal time $dt=ad\tau$. We therefore have a simpler Dirac equation

$$\gamma^0\chi'+\gamma^i\partial_i\chi+ma\chi=0 \tag{3.19}$$

Since the background is spatially translation invariant, the solution can be written in mode function as

$$\psi(\mathbf{x},t) \equiv a^{-\frac{3}{2}}(t)\chi(\mathbf{x},t) \tag{3.20}$$

$$= \frac{1}{a^{\frac{3}{2}}(t)}\int d^3q\sum_s[e^{i\mathbf{q}\cdot\mathbf{x}}\alpha(\mathbf{q},s)u_{\mathbf{q},s}(t)+e^{-i\mathbf{q}\cdot\mathbf{x}}\beta^{\dagger}(\mathbf{q},s)v_{\mathbf{q},s}(t)] \tag{3.21}$$

Similarly,

$$\bar{\psi}(\mathbf{x},t) \equiv a^{-\frac{3}{2}}(t)\bar{\chi}(\mathbf{x},t) \tag{3.22}$$

$$= \frac{1}{a^{\frac{3}{2}}(t)}\int d^3q \sum_s [e^{-i\mathbf{q}\cdot\mathbf{x}}\alpha^\dagger(\mathbf{q},s)\bar{u}_{\mathbf{q},s}(t) + e^{i\mathbf{q}\cdot\mathbf{x}}\beta(\mathbf{q},s)\bar{v}_{\mathbf{q},s}(t)] \tag{3.23}$$

where $s = \pm\frac{1}{2}$ stands for the spin and $u_{\mathbf{q},s}(t)$ and $v_{\mathbf{q},s}(t)$ satisfy

$$\gamma^0 u'_{\mathbf{q},s} + i\gamma^i q_i u_{\mathbf{q},s} + imau_{\mathbf{q},s} = 0 \tag{3.24}$$

and

$$\gamma^0 v'_{\mathbf{q},s} - i\gamma^i q_i v_{\mathbf{q},s} + imav_{\mathbf{q},s} = 0 \tag{3.25}$$

The creation and annihilation operators satisfy the anti-commutation relations

$$\Big\{\alpha(\mathbf{q},s),\alpha(\mathbf{q}',s')\Big\} = \Big\{\beta(\mathbf{q},s),\beta(\mathbf{q}',s')\Big\} = 0 \tag{3.26}$$

$$\Big\{\alpha(\mathbf{q},s),\alpha^\dagger(\mathbf{q}',s')\Big\} = \Big\{\beta(\mathbf{q},s),\beta^\dagger(\mathbf{q}',s')\Big\} = \delta_{ss'}\delta^3(\mathbf{q}-\mathbf{q}') \tag{3.27}$$

The gravitational field $\zeta(\mathbf{x},t)$ can be written in Fourier decomposition as

$$\zeta(\mathbf{x},t) = \int d^3q[e^{i\mathbf{q}\cdot\mathbf{x}}\alpha(\mathbf{q})\zeta_q(t) + e^{-i\mathbf{q}\cdot\mathbf{x}}\alpha^*(\mathbf{q})\zeta_q^*(t)] \tag{3.28}$$

where $\zeta_q(t)$ satisfies Mukhanov's equation [1] as [2]

$$\ddot{\zeta}_q + \frac{d}{dt}(\ln a^3\epsilon)\dot{\zeta}_q + \frac{q^2}{a^2}\zeta_q = 0 \tag{3.29}$$

[2]We work with interaction picture free field equations here to obtain time dependent propagators via their solutions. When loop effect is included, Mukhanov's equation is modified by varying the loop quantum effective action with respect to ζ.

Note that $\alpha(\mathbf{q})$ satisfies the commutation relations

$$[\alpha(\mathbf{q}), \alpha^*(\mathbf{q}')] = \delta^3(\mathbf{q} - \mathbf{q}') \tag{3.30}$$

$$[\alpha(\mathbf{q}), \alpha(\mathbf{q}')] = 0 \tag{3.31}$$

3.3 Fermion and Gravity Interaction Vertices

We can expand matter and gravity fluctuations in the actions of (3.1) beyond quadratic order in cosmological fluctuation. The cubic term and higher order terms are time dependent vertices that are needed in calculating loop diagrams. However, it is not necessary (at least, for the trilinear vertices) to directly expand gravitational field to find the interaction Hamiltonian of matter and gravity which may be complicated and contain many terms. The reason is that many terms are cancelled by the Bianchi Identity $T^{\mu\nu}_{;\nu} = 0$ and Mukhanov's free field equation in the interaction picture. As shown in appendix, all we need to do is to find the fermion energy momentum tensor to the quadratic order, which is simpler than expanding the full ADM fermionic action. Let us recall that the trilinear interaction of any general matter with zero expectation value is [2]

$$H_{\zeta MM}(t) = -\int d^3x \epsilon H a^5 (T^{00} + a^2 T^{ii}) \nabla^{-2}\dot{\zeta} + \dot{Y}(t) \tag{3.32}$$

where

$$Y(t) = a^6 T^{00} \left(\frac{\zeta}{Ha^3} - \frac{\epsilon}{a}\nabla^{-2}\dot{\zeta}\right) \tag{3.33}$$

and the term $\dot{Y}(t)$ can be removed by field redefinition of ζ as mentioned in [2]. The fermion energy momentum tensor in the presence of gravity is

$$T^{\mu\nu}_f = -\frac{1}{2}(\bar{\psi}^{\mu}_{;}\gamma^{\nu}\psi + \bar{\psi}^{\nu}_{;}\gamma^{\mu}\psi - \bar{\psi}\gamma^{\mu}\psi^{\nu}_{;} - \bar{\psi}\gamma^{\nu}\psi^{\mu}_{;}) \tag{3.34}$$

Therefore, we have the time component of energy momentum tensor to the quadratic order as

$$T_f^{00} = \dot{\bar{\psi}}\gamma^0\psi - \bar{\psi}\gamma^0\dot{\psi} \tag{3.35}$$

$$= \dot{\psi}^\dagger\beta\gamma^0\psi - \psi^\dagger\beta\gamma^0\dot{\psi} \tag{3.36}$$

$$= (-i)\Big(\dot{\psi}^\dagger\psi - \psi^\dagger\dot{\psi}\Big) \tag{3.37}$$

where $\bar{\psi} \equiv \psi^\dagger\beta$, $\beta \equiv i\gamma^0$, $(\gamma^0)^2 = -1$, and $\partial^0 = -\partial_0 = -\frac{\partial}{\partial t}$. Similarly, the spatial component of energy momentum tensor is

$$a^2 T_f^{ii} = \frac{1}{a}\Big(\bar{\psi}\gamma^i(\partial_i\psi) - (\partial_i\bar{\psi})\gamma^i\psi\Big) \tag{3.38}$$

where $\partial^i = \frac{\partial_i}{a^2}$ and $\gamma^\mu = V^\mu_\alpha\gamma^\alpha$. Therefore, $\gamma^0 = V^0_0\gamma^0 = \gamma^0$ and $\gamma^i = V^i_m\gamma^m = \frac{\delta^i_m\gamma^m}{a}$. These give us the Hamiltonian interaction of fermion and gravity to the cubic order as

$$H_{\zeta\bar{\psi}\psi}(t) = -\int d^3x\epsilon H a^5\Big[\dot{\bar{\psi}}\gamma^0\psi - \bar{\psi}\gamma^0\dot{\psi} \tag{3.39}$$

$$+\frac{1}{a}\Big(\bar{\psi}\gamma^i(\partial_i\psi) - (\partial_i\bar{\psi})\gamma^i\psi\Big)\Big]\nabla^{-2}\dot{\zeta} \tag{3.40}$$

Note that the interaction above is real and also valid for massive fermion.

3.4 Infrared Safe

Understanding what happened to the correlation of ζ at late time during inflation is crucial because we could connect this quantity with the observation. The question that concerns us is whether there is any infrared divergence when the quantum effect is taken into account. By infrared divergence we mean that the time or momentum integrals inside the loop blow up when taking $t \to \infty$ or $p \to 0$, respectively. This situation may or may not arise depending on the conditions of the theories. Before we proceed to the actual calculation, we first like to apply Weinberg's theorem [2] to fermionic fields to check if the time integral diverges at late time.

Let us first briefly review what Weinberg's theorem is. The theorem states that

"The integrals over time coordinates of interactions converge exponentially for $t \to \infty$, all interactions are either safe (a number of factors of $a(t)$ strictly less than $2\nu - 2$ for $2\nu \to 3$ in 3 space dimension) or dangerous (which grow at late time no faster than a and contain only fields, not time derivatives of fields) interactions"

We therefore need to know what the powers of a for ψ and $\dot{\psi}$ are before we can see the late time behavior of the final result. This can be done by solving Dirac's equation in FRW and see its behavior at late time. Fermion is somewhat different from scalar field in that the latter's equation of motion is second order and has two independent solutions. One approaches constant and another approaches a^{-3} at late time. Therefore, the scalar field σ is counted as at most a^0 or constant in the interaction. The time derivative of the scalar field $\dot{\sigma}$ is counted as a^{-2}[2]. It is only natural to ask

What are the powers of a in other fields and their time derivatives?

This can be answered by solving Dirac equations and calculating late time behavior in unequal time anti-commutation. To implement dimensional regularization, we replace $2\nu \to 3$ in Dirac's equation and hence,

$$\frac{1}{a^{\nu}}\frac{d}{dt}\Big(a^{\nu}\gamma^0\psi_q\Big) + \Big(\frac{i\gamma^i q_i}{a}\Big)\psi_q = 0 \tag{3.41}$$

We can solve this equation perturbatively, with $\frac{q}{a} \ll H$ at outside horizon,

$$\frac{1}{a^\nu}\frac{d}{dt}\Big(a^\nu\gamma^0\psi_q^{(n+1)}\Big) = -\Big(\frac{i\gamma^i q_i}{a}\Big)\psi_q^{(n)} \tag{3.42}$$

where $n = 0, 1, 2, ...$

To the 0^{th} order, $n = 0$, we have

$$\frac{1}{a^\nu}\frac{d}{dt}\Big(a^\nu\gamma^0\psi_q^{(1)}\Big) = 0 \tag{3.43}$$

or

$$\psi_q^{(1)}(t) = -\frac{\psi_q^o\gamma^0}{a^\nu} \tag{3.44}$$

To the 1^{th} order, $n = 1$, we have

$$\frac{1}{a^\nu}\frac{d}{dt}\Big(a^\nu\gamma^0\psi_q^{(2)}\Big) \quad = \quad -\Big(\frac{i\gamma^i q_i}{a}\Big)\psi_q^{(1)} \tag{3.45}$$

or

$$\psi_q^{(2)}(t) = -\frac{\psi_q^o\gamma^0}{a^\nu(t)}\Big(1 + iq_i\gamma^i\gamma^0\int_{-\infty}^{t}\frac{dt'}{a(t')}\Big) \tag{3.46}$$

We can proceed to an arbitrary order of n for higher q,

$$\psi_q(t) \to -\frac{\psi_q^o\gamma^0}{a^\nu(t)}\Big(1 + iq_i\gamma^i\gamma^0\int_{-\infty}^{t}\frac{dt'}{a(t')} - q_iq_j\gamma^i\gamma^0\gamma^j\gamma^0\int_{-\infty}^{t}\frac{dt'}{a(t')}\int_{-\infty}^{t'}\frac{dt''}{a(t'')} + ...\Big) \tag{3.47}$$

where ψ_q^o is a constant. We see that the massless fermionic field and its time derivative go as

$$\psi_q \sim a^{-\nu}, \dot{\psi}_q \sim c_1\gamma^0 a^{-\nu} + c_2\gamma^i a^{-\nu-1} \tag{3.48}$$

Another way to see what the power of a in the field ψ is is to use current conservation.

We can also consider the fermion mass in Dirac's equation

$$\dot{\psi} + \frac{3H}{2}\psi - \gamma^0\gamma^i\frac{\partial_i\psi}{a} - m\gamma^0\psi = 0 \tag{3.49}$$

To implement dimensional regularization, we replace $2\nu \to 3$ in Dirac's equation and hence,

$$\frac{1}{a^\nu}\frac{d}{dt}\Big(a^\nu\gamma^0\psi_q\Big) + \Big(\frac{i\gamma^i q_i}{a} + m\Big)\psi_q = 0 \tag{3.50}$$

Notice that this looks like *half* equation of scalar field where

$$\frac{1}{a^{2\nu}}\frac{d}{dt}\Big(a^{2\nu}\dot{\sigma}_q\Big) + \Big(\frac{q^2}{a^2} + m^2\Big)\sigma_q = 0 \tag{3.51}$$

Since $(\gamma^0)^2 = -1$, applying $\bar{\psi}\gamma^0$ to the LHS of the equation (3.49) gives

$$\bar{\psi}\gamma^0\dot{\psi} + \frac{3H}{2}\bar{\psi}\gamma^0\psi + \bar{\psi}\gamma^i\frac{\partial_i\psi}{a} + m\bar{\psi}\psi = 0 \tag{3.52}$$

Conjugation of Dirac equation in (3.49) is

$$\dot{\psi}^\dagger + \frac{3H}{2}\psi^\dagger - \frac{\partial_i\psi^\dagger}{a}\gamma^0\gamma^i + m\psi^\dagger\gamma^0 = 0 \tag{3.53}$$

where $(\gamma^0\gamma^i\partial_i\psi)^\dagger = (\partial_i\psi^\dagger)\gamma^0\gamma^i$[3]. Applying $\beta\gamma^0\psi$ to the RHS of the equation above, we have

$$\dot{\bar{\psi}}\gamma^0\psi + \frac{3H}{2}\bar{\psi}\gamma^0\psi + \frac{\partial_i\bar{\psi}}{a}\gamma^i\psi - m\bar{\psi}\psi = 0 \tag{3.54}$$

Combination of equations (3.52) and (3.54) *cancels* the mass term. We obtain the

[3] To see this, we use $\beta\gamma^{\mu\dagger}\beta = -\gamma^\mu$ and $\beta = i\gamma^0$. Hence, $(\gamma^0\gamma^i\partial_i\psi)^\dagger = -(\partial_i\psi)^\dagger(\gamma^i)^\dagger\gamma^0 = (\partial_i\psi)^\dagger\beta(\gamma^i)\beta\gamma^0 = (\partial_i\psi)^\dagger\gamma^0\gamma^i$

current conservation in expanding universe as

$$\frac{1}{a^3}\frac{d}{dt}(a^3\bar{\psi}\gamma^0\psi) + \frac{1}{a}\partial_i(\bar{\psi}\gamma^i\psi) = 0 \tag{3.55}$$

Hence, at zero momentum,

$$\psi^\dagger\psi \sim a^{-3} \tag{3.56}$$

Therefore, we expect the time integral to converge at late time. This can be seen through the interaction Hamiltonian of fermion and gravity ζ. Note that the energy momentum tensor of fermion is

$$T^{00} = \dot{\bar{\psi}}\gamma^0\psi - \bar{\psi}\gamma^0\dot{\psi} \tag{3.57}$$

and

$$a^2T^{ii} = \frac{1}{a}\Big(\bar{\psi}\gamma^i(\partial_i\psi) - (\partial_i\bar{\psi})\gamma^i\psi\Big) \tag{3.58}$$

Subtracting eqs. (3.54) and (3.52), we have

$$\dot{\bar{\psi}}\gamma^0\psi - \bar{\psi}\gamma^0\dot{\psi} + \frac{\partial_i\bar{\psi}}{a}\gamma^i\psi - \bar{\psi}\gamma^i\frac{\partial_i\psi}{a} - 2m\bar{\psi}\psi = 0 \tag{3.59}$$

Hence,

$$\begin{aligned} T^{00} + a^2T^{ii} &= \frac{2}{a}\Big(\bar{\psi}\gamma^i(\partial_i\psi) - (\partial_i\bar{\psi})\gamma^i\psi + ma\bar{\psi}\psi\Big) && (3.60) \\ &= -2\Big(\bar{\psi}\gamma^0\dot{\psi} - \dot{\bar{\psi}}\gamma^0\psi + m\bar{\psi}\psi\Big) && (3.61) \end{aligned}$$

Therefore, the trilinear interaction Hamiltonian is

$$\begin{aligned} H_{\zeta\bar{\psi}\psi}(t) &= 2\int d^3x\epsilon Ha^4\Big(\bar{\psi}\gamma^i(\partial_i\psi) - (\partial_i\bar{\psi})\gamma^i\psi + ma\bar{\psi}\psi\Big)\nabla^{-2}\dot{\zeta} && (3.62) \\ &= -2\int d^3x\epsilon Ha^5\Big(\bar{\psi}\gamma^0\dot{\psi} - \dot{\bar{\psi}}\gamma^0\psi + m\bar{\psi}\psi\Big)\nabla^{-2}\dot{\zeta} && (3.63) \end{aligned}$$

In $2\nu = 3$ space dimension, eq. (3.48) gives $\psi \sim a^{-\frac{3}{2}}$, and $\dot{\psi} \sim c_1\gamma^0 a^{-\frac{3}{2}} + c_2\gamma^i a^{-\frac{5}{2}}$. Since $\dot{\zeta} \sim a^{-2}$[2], the fermionic and gravity interactions in eqs. (3.62) and (3.60) are therefore *safe* interactions. The reasons [4] are that the first two terms go as $a^{5-2-1-2(\frac{3}{2})} = a^{5-2-\frac{3}{2}-\frac{5}{2}} = a^{-1}$. The third term, which is the mass term, goes as $a^{5-2-2(\frac{3}{2})} = a^0 =$ constant. We therefore expect the time integrals to converge at late time. It should be noted that the fermion mass term may be the most dominating among all terms as indicated by the power counting of a. The mass term in the interaction Hamiltonian approaches constant (after including factors of $\sqrt{-g}$ and ζ), rather than decaying away as a^{-1} in the derivative terms. Therefore, after integrating over time, the mass term will contribute $(\ln a)^n$ in the late time behavior for n points cosmological correlation functions.

3.5 Massless Fermion Result

We see that Dirac's equations in expanding universe (3.24) and (3.25) at $m = 0$ are the same as those in Minkowski space except that the physical time t is replaced with conformal time τ. So we can expect the plane wave solutions for $u_{\mathbf{q},s}(t)$ and $v_{\mathbf{q},s}(t)$ to be

$$u_{\mathbf{q},s}(t) = u^o_{\mathbf{q},s} e^{-iq\tau} \tag{3.64}$$

$$v_{\mathbf{q},s}(t) = v^o_{\mathbf{q},s} e^{iq\tau} \tag{3.65}$$

where $u^o_{\mathbf{q},s}$ and $v^o_{\mathbf{q},s}$ stand for constant coefficients at outside horizon. These coefficients can be determined by matching the solutions at deep inside horizon with those of flat space solutions. At deep inside horizon, the field does not feel the effect of the expansion of the universe. Therefore, the normalization factor can be chosen

[4]The terms involving $c_1\gamma^0 a^{-\frac{3}{2}}$ in $\dot{\psi}_q$ are cancelled in gravitational and fermionic interactions

in the same way as in Minkowski space

$$\sum_s u^o_{q,s}\bar{u}^o_{q,s} = \sum_s v^o_{q,s}\bar{v}^o_{q,s} = -\frac{i\gamma^\mu q_\mu}{2(2\pi)^3 q} \tag{3.66}$$

where $q \equiv q^0 = \sqrt{\mathbf{q}^2}$. We see that the momentum dependence q of the expectation value of fermion and anti-fermion pair $\langle\bar{\psi}\psi\rangle$ is far from the scale invariant spectrum q^{-3} and unlike that in the scalar case. However, this does not rule out that fermion could not seed the large scale structure of the universe observed today. The reason is that we never observe the product of either scalar or fermionic field but rather the product of temperature $\langle\frac{\delta T}{T}\frac{\delta T}{T}\rangle$ or density $\langle\frac{\delta\rho}{\rho}\frac{\delta\rho}{\rho}\rangle$ fluctuations, which are related to the conserved quantities $\langle\zeta\zeta\rangle$. Since ζ is purely a geometric quantity in the $\delta\varphi = \delta u = 0$ gauge and fermion interacts with the gravitational fluctuation, we therefore calculate how fermion affects ζ at the loop quantum level.
We now calculate the one-loop graph with two vertices of the two point function. Owing to the interaction of gravity and fermionic field, the quantum correction of the ζ spectrum is

$$\Big\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle_{loop} = -\int_{-\infty}^{t} dt_2 \int_{-\infty}^{t_2} dt_1 \Big\langle[H(t_1),[H(t_2),\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)]]\Big\rangle \tag{3.67}$$

As seen from the previous section, the trilinear interaction of massless fermion and gravity is

$$H_{\zeta\bar{\psi}\psi}(t) = -2\int d^3x \epsilon H a^4\Big(\bar{\psi}\gamma^i(\partial_i\psi) - (\partial_i\bar{\psi})\gamma^i\psi\Big)\nabla^{-2}\dot{\zeta} + \dot{Y}(t) \tag{3.68}$$

By counting the power of a, the interaction Hamiltonian goes as $a^{4-3-2} = a^{-1}$. Therefore, we expect this will be *infrared safe* according to Weinberg's theorem such that there should be no divergence in the time integrals when taking the limit $\tau \to 0$.

To quantize, we write ψ and $\bar{\psi}$ in terms of creation and annihilation operators in momentum space. We can use the formula (2.46) and (2.52) derived in the previous chapter. To match with the interaction Hamiltonian in eq. (3.68), we replace the interaction in eq. (2.25) with

$$\psi^*\psi \quad \rightarrow \quad \Big(\bar{\psi}\gamma^i \frac{\partial_i}{a}\psi - (\frac{\partial_i}{a}\bar{\psi})\gamma^i\psi\Big) \tag{3.69}$$

$$\zeta_q(t_{1,2}) \quad \rightarrow \quad -\dot{\zeta}_q(t_{1,2})/q^2 \tag{3.70}$$

$$V(t) \quad = \quad -2\epsilon H a^5(t) \tag{3.71}$$

Hence, $\mathcal{Z}$ in eq. (2.48) changes to

$$\mathcal{Z} \rightarrow \frac{1}{q^4}\Big(\dot{\zeta}_q(t_1)\zeta_q^*(t)(\dot{\zeta}_q(t_2)\zeta_q^*(t) - \zeta_q(t)\dot{\zeta}_q^*(t_2))\Big) \tag{3.72}$$

and $\mathcal{M}_\psi$ in eq. (2.52) changes to be

$$\mathcal{M}_\psi \rightarrow -\frac{1}{a_1 a_2}(p_i - p'_i)(p_j - p'_j)\sum_{r,s}\gamma^i U_{\mathbf{p},s}(t_1)\bar{U}_{\mathbf{p},s}(t_2)\gamma^j V_{\mathbf{p}',r}(t_2)\bar{V}_{\mathbf{p}',r}(t_1) \tag{3.73}$$

The equation above shows the need to sum over spin at different time. Fortunately, massless fermion is conformally flat so we still can use the spin sum formula from flat space. As seen from Dirac's equation, the solutions of massless fermion are just plane waves with conformal time, $u_{\mathbf{p},s}(\tau) = u^o_{\mathbf{p},s}e^{-ip\tau}$, after re-scaling the field such that $U_{\mathbf{p},s} = a^{-\frac{3}{2}}(t)u_{\mathbf{p},s}(t)$. The momentum dependent coefficient $u^o_{\mathbf{p},s}$ satisfies the spin sum formula [5]

$$\sum_s u^o_{p,s}\bar{u}^o_{p,s} = \sum_s v^o_{p,s}\bar{v}^o_{p,s} = -\frac{i\gamma^\alpha p_\alpha}{2(2\pi)^3 p} \tag{3.74}$$

[5] We emphasize that this formula is *only valid for massless fermion* since the mode solution of massless fermion is the same as that in flat space except we replace physical time t with conformal time τ and the spin sum is *time independent*. The situation is entirely different for massive fermion, as will be shown in the next section.

With the spin sum equation above, eq. (3.73) becomes

$$\mathcal{M}_\psi = (p_i - p'_i)(p_j - p'_j)\frac{p_\alpha p'_\beta}{4(2\pi)^6 pp' a_1^4 a_2^4} tr(\gamma^i\gamma^\alpha\gamma^j\gamma^\beta) e^{-i(p+p')(\tau_1-\tau_2)} \tag{3.75}$$

Since

$$tr(\gamma^i\gamma^\alpha\gamma^j\gamma^\beta) = 4(\eta^{i\alpha}\eta^{j\beta} - \eta^{ij}\eta^{\alpha\beta} + \eta^{i\beta}\eta^{\alpha j}) \tag{3.76}$$

Therefore,

$$\mathcal{M}_\psi = \frac{1}{(2\pi)^6 a_1^4 a_2^4}(p-p')^2(1+\hat{p}\cdot\hat{p}')e^{-i(p+p')(\tau_1-\tau_2)} \tag{3.77}$$

Substituting $\mathcal{M}_\psi$ into eq. (2.46) with $V(t) = -2\epsilon H a^5(t)$, we have

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x'})}\Big\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle_{loop} = -16(2\pi)^3\int d^3p d^3p' \delta^3(\mathbf{q}+\mathbf{p}-\mathbf{p}') \tag{3.78}$$

$$(p-p')^2(1+\hat{p}\cdot\hat{p}')\int_{-\infty}^{t} dt_2\epsilon_2 H_2 a_2\int_{-\infty}^{t_2} dt_1\epsilon_1 H_1 a_1 Re\Big(\mathcal{Z}e^{-i(p+p')(\tau_1-\tau_2)}\Big) \tag{3.79}$$

where $\mathcal{Z}$ is the contribution of the ζ part in eq. (3.72). To calculate $\mathcal{Z}$, we use the solution of free field Mukhanov's equation in interaction picture

$$\zeta_q(t) = \zeta_q^o e^{-iq\tau}(1+iq\tau) \tag{3.80}$$

where

$$|\zeta_q^o|^2 = \frac{8\pi G H^2(t_q)}{2(2\pi)^3\epsilon(t_q)q^3} \tag{3.81}$$

Hence,

$$\mathcal{Z} = \frac{|\zeta_q^o|^4}{H_1 H_2 a_1^2 a_2^2}\Big(e^{2iq\tau - iq(\tau_1+\tau_2)} - e^{iq(\tau_2-\tau_1)}\Big) \tag{3.82}$$

During slow roll inflation, we approximate $\epsilon_1 \approx \epsilon_2 \approx \epsilon(t_q)$. Integrating over conformal time τ_1, we get

$$\int d^3xe^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}\Big\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle_{loop} = -16(2\pi)^3|\zeta_q^o|^4\epsilon^2(t_q)\int d^3pd^3p'\delta^3(\mathbf{q}+\mathbf{p}+\mathbf{p}') \tag{3.83}$$

$$\times(p-p')^2(1+\hat{p}\cdot\hat{p}')Re\int_{-\infty}^{0} d\tau_2\frac{i}{q+p+p'}(e^{-2iq\tau_2}-1) \tag{3.84}$$

where an upper limit $t\to\infty$ or $\tau\to 0$ means a time during inflation but sufficiently late so that $a(t)$ is many e-folding larger than its value when $\frac{q}{a}$ falls below H. Integrating over conformal time τ_2 gives

$$Re\int_{-\infty}^{0} d\tau_2\frac{i}{q+p-p'}(e^{-2iq\tau_2}-1) = -\frac{1}{2q(q+p+p')} \tag{3.85}$$

Substituting eqs. (3.81) and (3.85) into eq. (3.83), we have

$$\int d^3xe^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}\Big\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle_{loop} = \frac{2(8\pi GH^2(t_q))^2}{(2\pi)^3q^7} \tag{3.86}$$

$$\times\int d^3pd^3p'\delta^3(\mathbf{q}+\mathbf{p}+\mathbf{p}')\frac{(p-p')^2}{q+p+p'}(1+\hat{p}\cdot\hat{p}') \tag{3.87}$$

Power counting shows that the results of the momentum integrals p ,p' will go as q^4. If we use dimensional regularization to remove UV divergence, the finite part of $\langle\zeta\zeta\rangle$ for massless fermion loop will go as $q^{-3}\ln q$. To determine the coefficient of the finite part of the momentum integral above, we will follow the calculation as done in [2] for the scalar case. Note that eq. (3.86) can be written as

$$\int d^3xe^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}\Big\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle_{loop} = \frac{2(8\pi GH^2(t_q))^2}{(2\pi)^3q^7}\Big[\frac{2\pi}{q}\mathcal{J}(q)\Big] \tag{3.88}$$

where

$$\mathcal{J}(q) \equiv \int_0^\infty pdp\int_{|p-q|}^{|p+q|} p'dp'\frac{(p-p')^2}{q+p+p'}\Big(1+\frac{p'^2-p^2-q^2}{2pq}\Big) \tag{3.89}$$

With dimensional regularization, the UV divergence of the integral above for $\delta = 0$ gives the pole term as

$$\frac{2\pi}{q}\mathcal{J}(q) \Rightarrow q^{4+\delta}F(\delta) \tag{3.90}$$

$$F(\delta) \to \frac{F_0}{\delta} + F_1 \tag{3.91}$$

Therefore, in the limit $\delta = 0$,

$$\frac{2\pi}{q}\mathcal{J}(q) = q^4\Big[F_0 \ln q + L\Big] \tag{3.92}$$

where L is a divergent constant. To eliminate divergence in the momentum integral above, we use Mathematica to integrate over p' and differentiate $\mathcal{J}(q)$ six times. Hence,

$$\frac{d^{(6)}\mathcal{J}(q)}{dq^6} = -\frac{8}{q} \tag{3.93}$$

where we take the limit $q \to 0$ *after* integrating over p. Hence,

$$\mathcal{J}(q) = q^5\Big(-\frac{8}{5!}\ln q + L\Big) \tag{3.94}$$

or

$$F_0 = -\frac{2\pi}{15} \tag{3.95}$$

Substituting $\mathcal{J}(q)$ back into eq. (3.88), we have the finite part of correlation function as

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}\Big\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle_{loop} = -\frac{4\pi(8\pi GH^2(t_q))^2}{15(2\pi)^3q^3}\Big[\ln q + C\Big] \tag{3.96}$$

with C an unknown constant. Notice that we have the *same* sign as that in the massless scalar loop because we don't have the time order product of fermion pairs in eq. (2.2). The opposite sign of fermion loop only arises in in-out theory when we time order the product of fermion pairs to close the loop. Moreover the result in eq. (3.96) is smaller than the classical result by a factor of GH^2.

3.6 Massive Fermion Mode Solution

The calculation is much more difficult for massive fermion. This is because mode solution of massive fermion at arbitrary wavelength during inflation is not a simple plane wave like that in massless or flat space. We cannot use the trace technology normally done for spinor in flat space since the spin sum at different time cannot be written in the compact form of γ matrices. We therefore need to solve Dirac's equation in expanding universe during inflation and perform the spin sum at different time by multiplying the matrices. From the fermionic field equation in expanding universe eq. (3.17), we see that if fermion is massless or if it does not couple to inflaton, the equations above are the same as those in Minkowski space except that the physical time t is replaced with the conformal timeτ. Therefore, we can carry out the spin sum as we typically do in flat space. The situation is quite different if fermion has mass or if it couples to inflaton (i.e $m\bar{\psi}\psi = \varphi\bar{\psi}\psi$), which is the more general case. To be more specific, the spin sum of massive fermion in expanding universe cannot be easily realized by using the projection operators typically done for flat space such that

$$\sum_s u_{\mathbf{p},s}\bar{u}_{\mathbf{p},s} \neq \frac{-i\gamma^\alpha p_\alpha + ma(t)}{2(2\pi)^3 p^0} \tag{3.97}$$

$$\sum_s v_{\mathbf{p},s}\bar{v}_{\mathbf{p},s} \neq -\frac{i\gamma^\alpha p_\alpha + ma(t)}{2(2\pi)^3 p^0} \tag{3.98}$$

where $p^0 \equiv \sqrt{\mathbf{p}^2 + (ma)^2}$. The right hand side of the equations above explicitly depend on a specific time t, but not the left hand side which needs the spin sum at two different times t_1 and t_2. As we will prove below, the right hand side of the equation above is *only valid at equal time and deep inside horizon* where the solution is the same as that in flat space.

In this section, we first start from solving Dirac's equation in FRW and then use that solution to calculate the spin sum at different times valid to arbitrary wavelength.

How do we solve Dirac's equation in expanding universe? In flat space, we first solve the matrix equation with zero spatial momentum ($\vec{p} = 0$) and Lorentz boost the solution to get an explicit solution at non-zero momentum. In FRW, it is not clear whether we can Lorentz boost as that in flat space. To avoid this ambiguity, we directly solve Dirac's equation at a finite momentum instead of solving it at zero momentum and Lorentz boost.

Defining $u_{s,\mathbf{p}}(\tau) \equiv [u_{+,\mathbf{p}}(\tau)S, u_{-,\mathbf{p}}(\tau)S]^T$ and $v_{s,\mathbf{p}}(\tau) \equiv [v_{+,\mathbf{p}}(\tau)S, v_{-,\mathbf{p}}(\tau)S]^T$ where S are the two component eigenvectors of the helicity operators. We use the Dirac representation of gamma matrices.

$$\gamma^0 = (-i)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.99}$$

$$\gamma^i = (-i)\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{3.100}$$

Therefore, eq. (3.24) gives

$$u'_{+,\mathbf{p}} + i(\vec{\sigma} \cdot \vec{p})u_{-,\mathbf{p}} + imau_{+,\mathbf{p}} = 0 \tag{3.101}$$

$$u'_{-,\mathbf{p}} + i(\vec{\sigma} \cdot \vec{p})u_{+,\mathbf{p}} - imau_{-,\mathbf{p}} = 0 \tag{3.102}$$

We see that the equations above are first order coupled differential equations. To decouple, we differentiate those equations one more time. With some algebra, we get two uncoupled second-order differentials equations,

$$u''_{\pm,\mathbf{p}} + (\vec{p}^2 + (ma)^2 \pm i(am)')u_{\pm,\mathbf{p}} = 0 \tag{3.103}$$

where $(\vec{\sigma}\cdot\vec{p})^2 = (p_x^2 + p_y^2 + p_z^2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \vec{p}^2 1_{2\times 2} \equiv p^2$. Therefore[6], $p \equiv \sqrt{\vec{p}^2} = |\vec{p}|$. Similarly, for v, eq. (3.25) gives

$$v'_{+,\mathbf{p}} - i(\vec{p}\cdot\vec{\sigma})v_{-,\mathbf{p}} + imav_{+,\mathbf{p}} = 0 \tag{3.104}$$

$$v'_{-,\mathbf{p}} - i(\vec{p}\cdot\vec{\sigma})v_{+,\mathbf{p}} - imav_{-,\mathbf{p}} = 0 \tag{3.105}$$

$$v''_{\pm,\mathbf{p}} + (\vec{p}^2 + (ma)^2 \pm i(am)')v_{\pm,\mathbf{p}} = 0 \tag{3.106}$$

In general, m does not have to be a constant in time. If a fermion is coupled to an inflaton, it can be the effective time dependent mass (i.e. $m(t) = \varphi(t)$). Since we are now considering the de-sitter phase inflation, so $a = -\frac{1}{H\tau}$. Since H in de Sitter phase is constant, we will now assume that the mass $m = \varphi$ is constant since $\bar{\varphi}$ and H do not change very much during inflation. Therefore, equations above are solvable as

$$u''_{\pm,\mathbf{p}} + \Big(\vec{p}^2 + \frac{1}{\tau^2}(r^2 \pm ir)\Big)u_{\pm,\mathbf{p}} = 0 \tag{3.107}$$

[6]This notation p should not be confused with $p^0 \equiv \sqrt{\vec{p}^2 + (ma)^2} \neq p$. $p^0 = p$ only in the massless case.

The solutions are

$$u_{FRW}(\mathbf{p},s,t) \equiv \begin{pmatrix} u_{+,\mathbf{p}} \times S \\ u_{-,\mathbf{p}} \times S \end{pmatrix} \equiv \begin{pmatrix} u_{\mu,\mathbf{p}} \times S \\ (\hat{p}\cdot\vec{\sigma})u_{\bar{\mu},\mathbf{p}} \times S \end{pmatrix} \tag{3.108}$$

$$= \begin{pmatrix} c_{1,p}\sqrt{-\tau}H_{\mu}^{(1)}(-p\tau) \times S \\ (\hat{p}\cdot\vec{\sigma})c_{2,p}\sqrt{-\tau}H_{\bar{\mu}}^{(1)}(-p\tau) \times S \end{pmatrix} \tag{3.109}$$

where $\mu \equiv \frac{1}{2} - ir$, $\bar{\mu} \equiv \frac{1}{2} + ir$, and $r \equiv \frac{m}{H}$.

We choose the initial conditions so that the positive frequency mode solutions match with the flat space time solutions at deep inside horizon[7]

$$u_{flat} = \Big(\frac{E+m}{2(2\pi)^3 E}\Big)^{\frac{1}{2}} \begin{pmatrix} 1 \times S \\ \frac{\vec{p}\cdot\vec{\sigma}/a}{E+m} \times S \end{pmatrix} e^{i\int_t^\infty E(t')dt'} \tag{3.110}$$

where $E^2 \equiv m^2 + \frac{\vec{p}^2}{a^2}$, $S = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for spin up and $S = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for spin down. Note that $S^\dagger S = 1$. We can check that, in the massless limit, $u_{flat} = \frac{1}{\sqrt{2(2\pi)^3}} e^{-ip\tau}$, as expected.

Similarly, the solution of v satisfies the equation

$$v''_{\pm,\mathbf{p}} + \Big(\vec{p}^2 + \frac{1}{\tau^2}(r^2 \pm ir)\Big)v_{\pm,\mathbf{p}} = 0 \tag{3.111}$$

[7]Note that the asymptotic behavior of Hankel's function for large $x \equiv -p\tau$ is $H_\nu^{(1)}(x) \to \sqrt{\frac{2}{\pi x}} e^{ix - i\nu\pi/2 - i\pi/4}(1 + \frac{i}{2x}(\nu + \frac{1}{2})(\nu - \frac{1}{2}) + ...)$

The solutions are

$$v_{FRW}(\mathbf{p}, s, t) \equiv \begin{pmatrix} v_+ \times \tilde{S} \\ v_- \times \tilde{S} \end{pmatrix} \tag{3.112}$$

$$= \begin{pmatrix} (\hat{p}\cdot\vec{\sigma})c_{3,p}\sqrt{-\tau}H_{\mu}^{(2)}(-p\tau) \times \tilde{S} \\ c_{4,p}\sqrt{-\tau}H_{\bar{\mu}}^{(2)}(-p\tau) \times \tilde{S} \end{pmatrix} \tag{3.113}$$

where we choose the initial conditions so that the negative frequency mode solutions match with the flat space time solutions at deep inside horizon

$$v_{flat} = \Big(\frac{E+m}{2(2\pi)^3 E}\Big)^{\frac{1}{2}} \begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}/a}{E+m} \times \tilde{S} \\ 1 \times \tilde{S} \end{pmatrix} e^{-i\int_t^\infty E(t')dt'} \tag{3.114}$$

where $\tilde{S} = e^{i\alpha}\sigma_2 S$ and $\tilde{S}^\dagger \tilde{S} = 1$. We can check that, in the massless limit, $v_{flat} = \frac{1}{\sqrt{2(2\pi)^3}} e^{ip\tau}$, as expected.

To find the normalization coefficients $c_{1\ldots4}(p)$, we match the solutions of eqs. (3.108) with (3.110) and (3.112) with (3.114) by using the asymptotic property of Hankel's functions. Hence,

$$c_{1,p} = -c_{4,p} = \frac{i\sqrt{\pi p}}{2(2\pi)^{\frac{3}{2}}} e^{\frac{\pi m}{2H}} \tag{3.115}$$

$$c_{2,p} = -c_{3,p} = \frac{i\sqrt{\pi p}}{2(2\pi)^{\frac{3}{2}}} e^{-\frac{\pi m}{2H}} \tag{3.116}$$

In conclusion of this section, the solutions of Dirac's equation in expanding universe needed to determine time dependent propagator and spin sum at different times are

$$u_{FRW} \equiv \begin{pmatrix} u_+ \times S \\ u_- \times S \end{pmatrix} \equiv \begin{pmatrix} u_\mu \times S \\ (\hat{p}\cdot\vec{\sigma})u_{\bar{\mu}} \times S \end{pmatrix} \tag{3.117}$$

$$= \frac{i\sqrt{\pi p|\tau|}}{2(2\pi)^{\frac{3}{2}}} \begin{pmatrix} e^{\frac{\pi m}{2H}} H_{\mu}^{(1)}(-p\tau) \times S \\ (\hat{p}\cdot\vec{\sigma})e^{-\frac{\pi m}{2H}} H_{\bar{\mu}}^{(1)}(-p\tau) \times S \end{pmatrix} \tag{3.118}$$

where we use the property $H^{(2)}_{\mu}(x) = (H^{(1)}_{\bar{\mu}}(x))^*$

3.7 Spin Sum at Different Time: Time Dependent Propagators

As a requirement of the time dependent fermion propagator, it is necessary to sum over spin at different times in massive fermion. Since the massive mode solution is no longer just a plane wave as in the case of flat space, we therefore need to sum over the spin by multiplying the two 2 component matrices of the mode solutions.

3.7.1 Short Wavelength

We like to start with spin sum at short wavelength. The reasons are that the spin sum is less complicated at short wavelength than at general wavelength and it also allows us to check the correctness with the more familiar spin sum in flat space time. Once we are comfortable with the spin sum at different time in short wavelength limit, we can easily extend the result to the general wavelength limit in the next subsection.

From eq. (3.110), we have

$$\sum_\sigma u_{\mathbf{p},\sigma}(t_1)u^\dagger_{\mathbf{p},\sigma}(t_2) = u_{\mathbf{p},\uparrow}(t_1)u^\dagger_{\mathbf{p},\uparrow}(t_2) + u_{\mathbf{p},\downarrow}(t_1)u^\dagger_{\mathbf{p},\downarrow}(t_2) \tag{3.119}$$

$$= \sqrt{\frac{(E_1+m)(E_2+m)}{4(2\pi)^6 E_1E_2}}\, e^{i\int_{t_1}^{t_2} E(t')dt'} \times \tag{3.120}$$

$$\begin{pmatrix} 1 & 0 & \frac{p_z}{a_2(E_2+m)} & \frac{p_-}{a_2(E_2+m)} \\ 0 & 1 & \frac{p_+}{a_2(E_2+m)} & -\frac{p_z}{a_2(E_2+m)} \\ \frac{p_z}{a_1(E_1+m)} & \frac{p_-}{a_1(E_1+m)} & \frac{\vec{p}^2}{a_1a_2(E_1+m)(E_2+m)} & 0 \\ \frac{p_+}{a_1(E_1+m)} & -\frac{p_z}{a_1(E_1+m)} & 0 & \frac{\vec{p}^2}{a_1a_2(E_1+m)(E_2+m)} \end{pmatrix} \tag{3.121}$$

$$= \frac{1}{2(2\pi)^3\sqrt{E_1E_2}}\, e^{i\int_{t_1}^{t_2} E(t')dt'} \times \tag{3.122}$$

$$\begin{pmatrix} \sqrt{(E_1+m)(E_2+m)}1_{2\times2} & \sqrt{\frac{E_1+m}{E_2+m}}(\vec{\sigma}\cdot\vec{p}/a_2)_{2\times2} \\ \sqrt{\frac{E_2+m}{E_1+m}}(\vec{\sigma}\cdot\vec{p}/a_1)_{2\times2} & \sqrt{(E_1-m)(E_2-m)}1_{2\times2} \end{pmatrix} \tag{3.123}$$

$$= \frac{1}{2(2\pi)^3\sqrt{p_1^0p_2^0}}\, e^{i\int_{\tau_1}^{\tau_2} p^0(\tau')d\tau'} \times \tag{3.124}$$

$$\begin{pmatrix} \sqrt{(p_1^0+ma_1)(p_2^0+ma_2)}1_{2\times2} & \sqrt{\frac{p_1^0+ma_1}{p_2^0+ma_2}}(\vec{\sigma}\cdot\vec{p})_{2\times2} \\ \sqrt{\frac{p_2^0+ma_2}{p_1^0+ma_1}}(\vec{\sigma}\cdot\vec{p})_{2\times2} & \sqrt{(p_1^0-ma_1)(p_2^0-ma_2)}1_{2\times2} \end{pmatrix} \tag{3.125}$$

where $p^0(t) = a(t)E(t)$. We see that the equation for spin sum at different time is rather complicated even at deep inside horizon and cannot be written in a simple form of γ matrices as in the flat space. However, we can check that, at equal time limit $t_1 = t_2 \equiv t$, the spin sum in equation above can be simplified as

$$\sum_\sigma u_{\mathbf{p},\sigma}(t)u^\dagger_{\mathbf{p},\sigma}(t) = \frac{1}{2(2\pi)^3p^0}\begin{pmatrix} (p^0+ma)1_{2\times2} & (\vec{\sigma}\cdot\vec{p})_{2\times2} \\ (\vec{\sigma}\cdot\vec{p})_{2\times2} & (p^0-ma)1_{2\times2} \end{pmatrix} \tag{3.126}$$

$$= \frac{1}{2(2\pi)^3p^0}(-ip^0\gamma_0 - i\vec{p}\cdot\vec{\gamma} + ma(t))\beta_{DR} \tag{3.127}$$

$$= \frac{1}{2(2\pi)^3p^0}(-ip^\alpha\gamma_\alpha + ma(t))\beta_{DR} \tag{3.128}$$

For $m = 0$, the spin sum at different time gives the more familiar formula used for massless fermion in the previous section

$$\sum_\sigma u_{\mathbf{p},\sigma}(t_1)u^\dagger_{\mathbf{p},\sigma}(t_2) = \frac{1}{2(2\pi)^3\sqrt{p_1^0 p_2^0}} e^{ip(\tau_2-\tau_1)} \begin{pmatrix} \sqrt{p_1^0 p_2^0}1_{2\times 2} & \sqrt{\frac{p_1^0}{p_2^0}}(\vec{\sigma}\cdot\vec{p})_{2\times 2} \\ \sqrt{\frac{p_2^0}{p_1^0}}(\vec{\sigma}\cdot\vec{p})_{2\times 2} & \sqrt{p_1^0 p_2^0}1_{2\times 2} \end{pmatrix} \quad (3.129)$$

$$= \frac{1}{2(2\pi)^3 p^0} e^{ip(\tau_2-\tau_1)} \begin{pmatrix} p^0_{2\times 2} & (\vec{\sigma}\cdot\vec{p})_{2\times 2} \\ (\vec{\sigma}\cdot\vec{p})_{2\times 2} & p^0_{2\times 2} \end{pmatrix} \quad (3.130)$$

$$= \frac{-i}{2(2\pi)^3 p^0}(p^0\gamma_0 + \vec{p}\cdot\vec{\gamma})\beta_{DR} e^{ip(\tau_2-\tau_1)} \quad (3.131)$$

$$= \frac{-ip^\alpha\gamma_\alpha}{2(2\pi)^3 p^0}\beta_{DR} e^{ip(\tau_2-\tau_1)} \quad (3.132)$$

where $p_1^0 = p_2^0 \equiv p^0$ is time independent for the massless case only.

3.7.2 Long Wavelength

For long wavelength, we take the limit of $|p\tau| \to 0$ in eq. (3.117). Hence,

$$U_q \to \begin{pmatrix} \mathcal{D}_q a^{\lambda_-} \\ \mathcal{C}_q a^{\lambda_+} \end{pmatrix} \quad (3.133)$$

$$(3.134)$$

where

$$\lambda_\pm = -\frac{3}{2} \pm \frac{im}{H} \quad (3.135)$$

and

$$\mathcal{C}_q = \frac{e^{-\frac{\pi m}{2H}}}{4\pi \sin\bar{\mu}\pi\Gamma(1-\bar{\mu})}\left(\frac{2H}{q}\right)^{\frac{im}{H}} \quad (3.136)$$

$$\mathcal{D}_q = \frac{e^{\frac{\pi m}{2H}}}{4\pi \sin\mu\pi\Gamma(1-\mu)}\left(\frac{2H}{q}\right)^{-\frac{im}{H}} \quad (3.137)$$

$$u_+ = v_-^\dagger = u_\mu = \frac{\Gamma(\mu)}{(2\pi)^2} e^{\frac{\pi r}{2}} (\frac{x}{2})^{ir} \tag{3.138}$$

and

$$u_- = v_+^\dagger = (\hat{p}\cdot\vec{\sigma}) u_{\bar{\mu}} = (\hat{p}\cdot\vec{\sigma}) \frac{\Gamma(\bar{\mu})}{(2\pi)^2} e^{-\frac{\pi r}{2}} (\frac{x}{2})^{-ir} \tag{3.139}$$

where $r = \frac{m}{H}$ and $x = -p\tau$. Using the mode solution above, we can write down the spin sum at different time by multiplying the 2×2 block matrix. With $\hat{p}\cdot\vec{\sigma} = \hat{p}\cdot\vec{\sigma}^\dagger$, we have

$$\sum_\sigma u_{\mathbf{p},\sigma}(t_1) u^\dagger_{\mathbf{p},\sigma}(t_2) = \frac{1}{(2\pi)^4} \times \tag{3.140}$$

$$\begin{pmatrix} |\Gamma(\mu)|^2 e^{\pi r} (\frac{x_1}{x_2})^{ir} 1_{2\times 2} & (\Gamma(\mu))^2 (x_1 x_2)^{ir} (\vec{\sigma}\cdot\hat{p})_{2\times 2} \\ (\Gamma(\bar{\mu}))^2 (x_1 x_2)^{-ir} (\vec{\sigma}\cdot\hat{p})_{2\times 2} & |\Gamma(\mu)|^2 e^{-\pi r} (\frac{x_1}{x_2})^{-ir} 1_{2\times 2} \end{pmatrix} \tag{3.141}$$

for spin sum of u solution and

$$\sum_\sigma v_{\mathbf{p},\sigma}(t_1) v^\dagger_{\mathbf{p},\sigma}(t_2) = \frac{1}{(2\pi)^4} \times \tag{3.142}$$

$$\begin{pmatrix} |\Gamma(\mu)|^2 e^{-\pi r} (\frac{x_1}{x_2})^{-ir} 1_{2\times 2} & (\Gamma(\bar{\mu}))^2 (x_1 x_2)^{-ir} (\vec{\sigma}\cdot\hat{p})_{2\times 2} \\ (\Gamma(\mu))^2 (x_1 x_2)^{ir} (\vec{\sigma}\cdot\hat{p})_{2\times 2} & |\Gamma(\mu)|^2 e^{\pi r} (\frac{x_1}{x_2})^{ir} 1_{2\times 2} \end{pmatrix} \tag{3.143}$$

for spin sum of v solution.

Notice that the mode solutions $u_{\mathbf{p},s}(t)$ and $v_{-\mathbf{p},r}(t)$ satisfy the orthogonal relation

$$v^\dagger_{-\mathbf{p},r}(t) u_{\mathbf{p},s}(t) = u^\dagger_{-\mathbf{p},r}(t) v_{\mathbf{p},s}(t) = 0 \tag{3.144}$$

and

$$u^\dagger_{\mathbf{p},r}(t)u_{\mathbf{p},s}(t) = v^\dagger_{\mathbf{p},r}(t)v_{\mathbf{p},s}(t) = (|u_\mu(t)|^2 + |u_{\bar{\mu}}(t)|^2)\delta_{rs} \tag{3.145}$$

$$\rightarrow \frac{|\Gamma(\mu)|^2}{(2\pi)^4}\Big(e^{\pi r} + e^{-\pi r}\Big) \tag{3.146}$$

$$= \frac{1}{(2\pi)^3} \tag{3.147}$$

at late time and the scalar bilinear satisfies

$$\bar{u}_{\mathbf{p},r}(t)u_{\mathbf{p},s}(t) = \bar{v}_{\mathbf{p},r}(t)v_{\mathbf{p},s}(t) = (|u_\mu(t)|^2 - |u_{\bar{\mu}}(t)|^2)\delta_{rs} \tag{3.148}$$

$$\rightarrow \frac{|\Gamma(\mu)|^2}{(2\pi)^4}\Big(e^{\pi r} - e^{-\pi r}\Big) \tag{3.149}$$

$$= \frac{\tanh \pi r}{(2\pi)^3} \tag{3.150}$$

where we use $|\Gamma(\mu)|^2 = \frac{\pi}{\cosh \pi r}$

Since $\psi \equiv a^{-\frac{3}{2}}(u, v)$, we see from eq. (3.148) that the bilinear $a^3\bar{\psi}\psi$ or $a^3\psi^\dagger\psi$ approaches constant at long wavelength limit.

3.7.3 General Wavelength

We are now more comfortable in calculating the spin sum for general wavelength. From the solution of Dirac equation in dS-FRW (3.108), we can write the spin up in four matrix component as

$$u_{\mathbf{p},\uparrow} = \begin{pmatrix} u_+ \\ 0 \\ \hat{p}_z u_- \\ \hat{p}_+ u_- \end{pmatrix} = \frac{i\sqrt{\pi x}}{2(2\pi)^{\frac{3}{2}}} e^{-\frac{\pi r}{2}} \begin{pmatrix} e^{\pi r} H^{(1)}_\mu(x) \\ 0 \\ \hat{p}_z H^{(1)}_{\bar{\mu}}(x) \\ \hat{p}_+ H^{(1)}_{\bar{\mu}}(x) \end{pmatrix} \tag{3.151}$$

Similarly, for spin down solution,

$$u_{\mathbf{p},\downarrow} = \begin{pmatrix} 0 \\ u_+ \\ \hat{p}_- u_- \\ -\hat{p}_z u_- \end{pmatrix} = \frac{i\sqrt{\pi x}}{2(2\pi)^{\frac{3}{2}}} e^{-\frac{\pi r}{2}} \begin{pmatrix} 0 \\ e^{\pi r} H_{\mu}^{(1)}(x) \\ \hat{p}_- H_{\bar{\mu}}^{(1)}(x) \\ -\hat{p}_z H_{\bar{\mu}}^{(1)}(x) \end{pmatrix} \tag{3.152}$$

where $r \equiv \frac{m}{H}$ and $x \equiv -p\tau$. Using the equations above, the general formula of spin sum at different time and general wavelength is

$$\sum_{\sigma} u_{\mathbf{p},\sigma}(t_1) u^{\dagger}_{\mathbf{p},\sigma}(t_2) = u_{\mathbf{p},\uparrow}(t_1) u^{\dagger}_{\mathbf{p},\uparrow}(t_2) + u_{\mathbf{p},\downarrow}(t_1) u^{\dagger}_{\mathbf{p},\downarrow}(t_2) \tag{3.153}$$

$$= \begin{pmatrix} u_{\mu}(t_1) u^{*}_{\mu}(t_2) 1_{2\times 2} & u_{\mu}(t_1) u^{*}_{\bar{\mu}}(t_2) (\vec{\sigma} \cdot \hat{p})_{2\times 2} \\ u_{\bar{\mu}}(t_1) u^{*}_{\mu}(t_2) (\vec{\sigma} \cdot \hat{p})_{2\times 2} & u_{\bar{\mu}}(t_1) u^{*}_{\bar{\mu}}(t_2) 1_{2\times 2} \end{pmatrix} \tag{3.154}$$

For the positive mode solution at arbitrary wavelength, we have

$$\sum_{\sigma} u_{\mathbf{p},\sigma}(t_1) u^{\dagger}_{\mathbf{p},\sigma}(t_2) = \frac{\pi\sqrt{x_1 x_2}}{4(2\pi)^3} \tag{3.155}$$

$$\begin{pmatrix} e^{\pi r} H^{(1)}_{\mu}(x_1) H^{*(1)}_{\mu}(x_2) 1_{2\times 2} & H^{(1)}_{\mu}(x_1) H^{*(1)}_{\bar{\mu}}(x_2) (\vec{\sigma} \cdot \hat{p})_{2\times 2} \\ H^{(1)}_{\bar{\mu}}(x_1) H^{*(1)}_{\mu}(x_2) (\vec{\sigma} \cdot \hat{p})_{2\times 2} & e^{-\pi r} H^{(1)}_{\bar{\mu}}(x_1) H^{*(1)}_{\bar{\mu}}(x_2) 1_{2\times 2} \end{pmatrix} \tag{3.156}$$

where $\mu \equiv \frac{1}{2} - ir$, $r \equiv \frac{m}{H}$, and $x \equiv -p\tau$

For negative frequency solutions v, we have for spin up

$$v_{\mathbf{p},\uparrow} = \begin{pmatrix} \hat{p}_z v_\mu \\ \hat{p}_+ v_\mu \\ v_{\bar{\mu}} \\ 0 \end{pmatrix} = -\frac{i\sqrt{\pi x}}{2(2\pi)^{\frac{3}{2}}} e^{-\frac{\pi r}{2}} \begin{pmatrix} \hat{p}_z H_\mu^{(2)}(x) \\ \hat{p}_+ H_\mu^{(2)}(x) \\ e^{\pi r} H_{\bar{\mu}}^{(2)}(x) \\ 0 \end{pmatrix} \tag{3.157}$$

and for spin down

$$v_{\mathbf{p},\downarrow} = \begin{pmatrix} \hat{p}_- v_\mu \\ -\hat{p}_z v_\mu \\ 0 \\ v_{\bar{\mu}} \end{pmatrix} = -\frac{i\sqrt{\pi x}}{2(2\pi)^{\frac{3}{2}}} e^{-\frac{\pi r}{2}} \begin{pmatrix} \hat{p}_- H_\mu^{(2)}(x) \\ -\hat{p}_z H_\mu^{(2)}(x) \\ 0 \\ e^{\pi r} H_{\bar{\mu}}^{(2)}(x) \end{pmatrix} \tag{3.158}$$

The formula of unequal time spin sum at general wavelength is

$$\sum_\sigma v_{\mathbf{p},\sigma}(t_1) v^\dagger_{\mathbf{p},\sigma}(t_2) = v_{\mathbf{p},\uparrow}(t_1) v^\dagger_{\mathbf{p},\uparrow}(t_2) + v_{\mathbf{p},\downarrow}(t_1) v^\dagger_{\mathbf{p},\downarrow}(t_2) \tag{3.159}$$

$$= \begin{pmatrix} \hat{p}^2 v_\mu(t_1) v_\mu^*(t_2) 1_{2\times 2} & v_\mu(t_1) v_{\bar{\mu}}^*(t_2) (\vec{\sigma}\cdot\hat{p})_{2\times 2} \\ v_{\bar{\mu}}(t_1) v_\mu^*(t_2) (\vec{\sigma}\cdot\hat{p})_{2\times 2} & v_{\bar{\mu}}(t_1) v_{\bar{\mu}}^*(t_2) 1_{2\times 2} \end{pmatrix} \tag{3.160}$$

For the negative mode solution at arbitrary wavelength, we have

$$\sum_\sigma v_{\mathbf{p},\sigma}(t_1) v^\dagger_{\mathbf{p},\sigma}(t_2) = \frac{\pi\sqrt{x_1 x_2}}{4(2\pi)^3} \tag{3.161}$$

$$\begin{pmatrix} e^{-\pi r} H_\mu^{(2)}(x_1) H_\mu^{*(2)}(x_2) 1_{2\times 2} & H_\mu^{(2)}(x_1) H_{\bar{\mu}}^{*(2)}(x_2) (\vec{\sigma}\cdot\hat{p})_{2\times 2} \\ H_{\bar{\mu}}^{(2)}(x_1) H_\mu^{*(2)}(x_2) (\vec{\sigma}\cdot\hat{p})_{2\times 2} & e^{\pi r} H_{\bar{\mu}}^{(2)}(x_1) H_{\bar{\mu}}^{*(2)}(x_2) 1_{2\times 2} \end{pmatrix} \tag{3.162}$$

In the short wavelength limit ($x \equiv -p\tau \to \infty$),

$$H_\mu^{(2)}(x) \simeq \sqrt{\frac{2}{\pi x}} e^{-i(x-\frac{\pi\mu}{2}-\frac{\pi}{4})} \Big(1 + \frac{1}{2ix}(\mu + \frac{1}{2})(\mu - \frac{1}{2}) + ...\Big) \tag{3.163}$$

Therefore,

$$\sum_\sigma v_{\mathbf{p},\sigma}(t_1) v^\dagger_{\mathbf{p},\sigma}(t_2) \simeq \frac{e^{ip(\tau_1-\tau_2)}}{2(2\pi)^3} \times \tag{3.164}$$

$$\begin{pmatrix} (1 - \frac{r(1-ir)}{2x_1} - \frac{r(1+ir)}{2x_2})1_{2\times 2} & (1 - \frac{r(1-ir)}{2x_1} + \frac{r(1-ir)}{2x_2})(\vec{\sigma} \cdot \hat{p})_{2\times 2} \\ (1 + \frac{r(1+ir)}{2x_1} - \frac{r(1+ir)}{2x_2})(\vec{\sigma} \cdot \hat{p})_{2\times 2} & (1 + \frac{r(1+ir)}{2x_1} + \frac{r(1-ir)}{2x_2})1_{2\times 2} \end{pmatrix} \tag{3.165}$$

where $\frac{r}{x} = -\frac{m}{Hp\tau} = \frac{ma}{p}$.

To check the spin sum above with that of v_{flat} solutions, we substitute eq. (3.114) into eq.(3.159). Hence,

$$\sum_\sigma v_{\mathbf{p},\sigma}(t_1) v^\dagger_{\mathbf{p},\sigma}(t_2) = \frac{e^{-i\int_{t_1}^{t_2} E(t')dt'}}{2(2\pi)^3} \sqrt{\frac{(p_1^0 + ma_1)(p_2^0 + ma_2)}{p_1^0 p_2^0}} \times \tag{3.166}$$

$$\begin{pmatrix} \frac{\vec{p}^2}{(p_1^0+ma_1)(p_2^0+ma_2)} 1_{2\times 2} & \frac{(\vec{\sigma}\cdot\vec{p})_{2\times 2}}{(p_1^0+ma_1)} \\ \frac{(\vec{\sigma}\cdot\vec{p})_{2\times 2}}{(p_2^0+ma_2)} & 1_{2\times 2} \end{pmatrix} \tag{3.167}$$

where $p = |\vec{p}| = \sqrt{(p^0)_1^2 - (ma_1)^2} = \sqrt{(p^0)_2^2 - (ma_2)^2}$, $p^0 = aE$. Therefore, we can write $\vec{p}^2 = \Big((p_1^0 - ma_1)(p_1^0 + ma_1)(p_2^0 - ma_2)(p_2^0 + ma_2)\Big)^{\frac{1}{2}}$ Hence, we see that eq. (3.164) is the correct approximation of eq. (3.166) at short wavelength.

Note that we can obtain simple spin sum of massive fermion (in terms of γ matrices) *only at equal time.* When $t_1 = t_2 \equiv t$, eq. (3.166) becomes

$$\sum_{\sigma} v_{\mathbf{p},\sigma}(t) v^{\dagger}_{\mathbf{p},\sigma}(t) = \frac{1}{2(2\pi)^3 p^0} \begin{pmatrix} (p^0 - ma)1_{2\times 2} & (\vec{\sigma}\cdot\vec{p})_{2\times 2} \\ (\vec{\sigma}\cdot\vec{p})_{2\times 2} & (p^0 + ma)1_{2\times 2} \end{pmatrix} \tag{3.168}$$

$$= \frac{1}{2(2\pi)^3 p^0}(-ip^0\gamma_0 - i\vec{p}\cdot\vec{\gamma} - ma)\beta_{DR} \tag{3.169}$$

$$= -\frac{1}{2(2\pi)^3 p^0}(ip^\alpha\gamma_\alpha + ma(t))\beta_{DR} \tag{3.170}$$

3.8 Massive Fermion Result

We now go to a more detail calculation of the integrand contributed by the massive fermion part $\mathcal{M}_\psi$.

We can use the formula (2.46) and (2.52) derived in chapter 2. To match with the more realistic interaction Hamiltonian that arises after expansion in eq. (3.62), we replace the interaction in eq. (2.25) with

$$\psi^*\psi \rightarrow \left(\bar{\psi}\gamma^0\dot{\psi} - \dot{\bar{\psi}}\gamma^0\psi + m\bar{\psi}\psi\right) \tag{3.171}$$

$$\zeta_q(t_{1,2}) \rightarrow -\dot{\zeta}_q(t_{1,2})/q^2 \tag{3.172}$$

$$V(t) \rightarrow -2\epsilon H a^5(t) \tag{3.173}$$

$\mathcal{Z}$ is still the same as that in eq. (3.72) for the massless case. With the re-scaling of $U_{\mathbf{p},s}(t) = a^{-\frac{3}{2}} u_{\mathbf{p},s}(t)$, $\mathcal{M}_\psi$ in eq. (2.52) becomes

$$\mathcal{M}_\psi \rightarrow -\frac{1}{a_1^3 a_2^3}\left(\mathcal{A} - m^2\mathcal{B} + im\mathcal{C} + im\mathcal{D}\right) \tag{3.174}$$

where $\mathcal{A}$ is the contribution from purely time derivative terms, B from the $\bar{\psi}\psi$ terms, $\mathcal{C}$ and $\mathcal{D}$ from the cross terms such that

$$\mathcal{A} = \sum_{r,s} \Big(v^{\dagger}_{p',r}(t_1)\dot{u}_{p,s}(t_1)u^{\dagger}_{p,s}(t_2)\dot{v}_{p',r}(t_2) + \dot{v}^{\dagger}_{p',r}(t_1)u_{p,s}(t_1)\dot{u}^{\dagger}_{p,s}(t_2)v_{p',r}(t_2) \tag{3.175}$$

$$-v^{\dagger}_{p',r}(t_1)\dot{u}_{p,s}(t_1)\dot{u}^{\dagger}_{p,s}(t_2)v_{p',r}(t_2) - \dot{v}^{\dagger}_{p',r}(t_1)u_{p,s}(t_1)u^{\dagger}_{p,s}(t_2)\dot{v}_{p',r}(t_2) \Big) \tag{3.176}$$

$$\mathcal{B} = \sum_{r,s} \bar{v}_{p',r}(t_1)u_{p,s}(t_1)\bar{u}_{p,s}(t_2)v_{p',r}(t_2) \tag{3.177}$$

$$\mathcal{C} = \sum_{r,s} \Big(v^{\dagger}_{p',r}(t_1)\dot{u}_{p,s}(t_1) - \dot{v}^{\dagger}_{p',r}(t_1)u_{p,s}(t_1) \Big) \bar{u}_{p,s}(t_2)v_{p',r}(t_2) \tag{3.178}$$

$$\mathcal{D} = \sum_{r,s} \bar{v}_{p',r}(t_1)u_{p,s}(t_1) \Big(u^{\dagger}_{p,s}(t_2)\dot{v}_{p',r}(t_2) - \dot{u}^{\dagger}_{p,s}(t_2)v_{p',r}(t_2) \Big) \tag{3.179}$$

As seen above, we need to know the spin sum at different time (corresponding to the time dependent propagators). Using eq. (3.153) with $v_{\mu} = u^*_{\bar{\mu}}$ and $v_{\bar{\mu}} = u^*_{\mu}$ in eq. (3.175), we have

$$\mathcal{A} = -2\Big[\Big(u^*_{\bar{\mu},p'}\dot{u}^*_{\mu,p} - \dot{u}^*_{\bar{\mu},p'}u^*_{\mu,p} \Big)|_{t_2} \Big((u_{\bar{\mu},p'}\dot{u}_{\mu,p} - \dot{u}_{\bar{\mu},p'}u_{\mu,p}) \tag{3.180}$$

$$+(\hat{p}\cdot\hat{p}')(u_{\mu,p'}\dot{u}_{\bar{\mu},p} - \dot{u}_{\mu,p'}u_{\bar{\mu},p}) \Big)|_{t_1} + \Big(\mu \leftrightarrow \bar{\mu} \Big) \Big] \tag{3.181}$$

where the $\mu \leftrightarrow \bar{\mu}$ terms arise when we take the trace in the lower component of the multiplied matrices. We can rewrite the result in a more compact form as

$$\mathcal{A} = -2\Big(\alpha^*_2\alpha_1 + \bar{\alpha}^*_2\bar{\alpha}_1 + (\hat{p}\cdot\hat{p}')(\alpha^*_2\bar{\alpha}_1 + \bar{\alpha}^*_2\alpha_1) \Big) \tag{3.182}$$

where

$$\alpha \equiv u_{\bar{\mu},p'}\dot{u}_{\mu,p} - \dot{u}_{\bar{\mu},p'}u_{\mu,p} \tag{3.183}$$

and

$$\bar{\alpha} \equiv u_{\mu,p'}\dot{u}_{\bar{\mu},p} - \dot{u}_{\mu,p'}u_{\bar{\mu},p} \tag{3.184}$$

We follow the same approach to calculate $\mathcal{B}$. With the spin sum equations in (3.153) with $v_\mu = u^*_{\bar{\mu}}$ and $v_{\bar{\mu}} = u^*_\mu$, we have

$$\begin{aligned}
\mathcal{B} &= \sum_{r,s} v_{\mathbf{p}',r}(t_2)v^\dagger_{\mathbf{p}',r}(t_1)\beta u_{\mathbf{p},s}(t_1)u^\dagger_{\mathbf{p},s}(t_2)\beta & (3.185)\\
&= tr\begin{pmatrix} u^*_{\bar{\mu},p'}(t_2)u_{\bar{\mu},p'}(t_1)1_{2\times2} & -u^*_{\bar{\mu},p'}(t_2)u_{\mu,p'}(t_1)(\vec{\sigma}\cdot\hat{p}')_{2\times2} \\ u^*_{\mu,p'}(t_2)u_{\bar{\mu},p'}(t_1)(\vec{\sigma}\cdot\hat{p}')_{2\times2} & -u^*_{\mu,p'}(t_2)u_{\mu,p'}(t_1)1_{2\times2} \end{pmatrix} & (3.186)\\
&\quad\times\begin{pmatrix} u_{\mu,p}(t_1)u^*_{\mu,p}(t_2)1_{2\times2} & -u_{\mu,p}(t_1)u^*_{\bar{\mu},p}(t_2)(\vec{\sigma}\cdot\hat{p})_{2\times2} \\ u_{\bar{\mu},p}(t_1)u^*_{\mu,p}(t_2)(\vec{\sigma}\cdot\hat{p})_{2\times2} & -u_{\bar{\mu},p}(t_1)u^*_{\bar{\mu},p}(t_2)1_{2\times2} \end{pmatrix} & (3.187)\\
&= 2\Big(u^*_{\bar{\mu},p'}(t_2)u_{\mu,p'}(t_1)u_{\mu,p}(t_1)u^*_{\mu,p}(t_2) - & (3.188)\\
&\quad(\hat{p}\cdot\hat{p}')u^*_{\bar{\mu},p'}(t_2)u_{\mu,p'}(t_1)u_{\bar{\mu},p}(t_1)u^*_{\mu,p}(t_2) + (\mu\leftrightarrow\bar{\mu})\Big) & (3.189)
\end{aligned}$$

Note that the "-" sign in front of $\hat{p}\cdot\hat{p}'$ arises due to $\bar{u} \equiv u^\dagger\beta$ and $\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The equation above can be written in a more compact form of

$$\mathcal{B} = 2\Big(\kappa^*_2\kappa_1 + \bar{\kappa}^*_2\bar{\kappa}_1 - (\hat{p}\cdot\hat{p}')(\kappa^*_2\bar{\kappa}_1 + \bar{\kappa}^*_2\kappa_1)\Big) \tag{3.190}$$

where

$$\kappa \equiv u_{\bar{\mu},p'}u_{\mu,p}, \bar{\kappa} \equiv u_{\mu,p'}u_{\bar{\mu},p} \tag{3.191}$$

To calculate $\mathcal{C}$, we have

$$\mathcal{C} = 2\Big(u^*_{\bar{\mu},p'}(t_2)u^*_{\mu,p}(t_2)(u_{\bar{\mu},p'}(t_1)\dot{u}_{\mu,p}(t_1) - \dot{u}_{\bar{\mu},p'}(t_1)u_{\mu,p}(t_1) \tag{3.192}$$

$$+(\hat{p}\cdot\hat{p}')(u_{\mu,p'}(t_1)\dot{u}_{\bar{\mu},p}(t_1) - \dot{u}_{\mu,p'}(t_1)u_{\bar{\mu},p}(t_1))) - (\mu \leftrightarrow \bar{\mu})\Big) \tag{3.193}$$

Hence, from eqs. (3.183) and (3.191),

$$\mathcal{C} = 2\Big(\kappa_2^*\alpha_1 - \bar{\kappa}_2^*\bar{\alpha}_1 + (\hat{p}\cdot\hat{p}')(\kappa_2^*\bar{\alpha}_1 - \bar{\kappa}_2^*\alpha_1)\Big) \tag{3.194}$$

Similarly, for $\mathcal{D}$,

$$\mathcal{D} = -2\Big(\alpha_2^*\kappa_1 - \bar{\alpha}_2^*\bar{\kappa}_1 + (\hat{p}\cdot\hat{p}')(\bar{\alpha}_2^*\kappa_1 - \alpha_2^*\bar{\kappa}_1)\Big) \tag{3.195}$$

Notice from eqs. (3.194) and (3.195) that

$$\mathcal{D} = -\mathcal{C}^*(t_1 \leftrightarrow t_2) \tag{3.196}$$

as expected from the fact that

$$\langle(\psi_1^\dagger\dot{\psi}_1 - \dot{\psi}_1^\dagger\psi_1)\bar{\psi}_2\psi_2\rangle_0^\dagger = -\langle\bar{\psi}_2\psi_2(\psi_1^\dagger\dot{\psi}_1 - \dot{\psi}_1^\dagger\psi_1)\rangle_0 \tag{3.197}$$

Substituting eqs. (3.182), (3.190), (3.194), and (3.195) into eq. (3.174), we have the result of fermionic part as

$$\mathcal{M}_\psi = \frac{2}{a_1^3a_2^3}\Big((\alpha_2^*\alpha_1 + \bar{\alpha}_2^*\bar{\alpha}_1 + (\hat{p}\cdot\hat{p}')(\alpha_2^*\bar{\alpha}_1 + \bar{\alpha}_2^*\alpha_1)) \tag{3.198}$$

$$+m^2(\kappa_2^*\kappa_1 + \bar{\kappa}_2^*\bar{\kappa}_1 - (\hat{p}\cdot\hat{p}')(\kappa_2^*\bar{\kappa}_1 + \bar{\kappa}_2^*\kappa_1)) \tag{3.199}$$

$$-im(\kappa_2^*\alpha_1 - \bar{\kappa}_2^*\bar{\alpha}_1 + (\hat{p}\cdot\hat{p}')(\kappa_2^*\bar{\alpha}_1 - \bar{\kappa}_2^*\alpha_1)) \tag{3.200}$$

$$+im(\alpha_2^*\kappa_1 - \bar{\alpha}_2^*\bar{\kappa}_1 + (\hat{p}\cdot\hat{p}')(\bar{\alpha}_2^*\kappa_1 - \alpha_2^*\bar{\kappa}_1))\Big) \tag{3.201}$$

The equation above can be written as $|\alpha \pm im\kappa|^2$, if all are treated at equal time. Nevertheless, even though the time can be different $t_1 \neq t_2$, we can simplify further as

$$\begin{aligned}
\mathcal{M}_\psi &= \frac{2}{a_1^3 a_2^3}\Big(\sigma_2^*\sigma_1 + \bar{\sigma}_2^*\bar{\sigma}_1 + (\hat{p}\cdot\hat{p}')(\sigma_2^*\bar{\sigma}_1 + \bar{\sigma}_2^*\sigma_1)\Big) && (3.202)\\
&= \frac{2}{a_1^3 a_2^3}(\hat{p}\sigma_2^* + \hat{p}'\bar{\sigma}_2^*)\cdot(\hat{p}\sigma_1 + \hat{p}'\bar{\sigma}_1) && (3.203)\\
&\equiv \frac{2}{a_1^3 a_2^3}\vec{\pi}_2^*\cdot\vec{\pi}_1 && (3.204)
\end{aligned}$$

where the σs are

$$\begin{aligned}
\sigma &\equiv \alpha + im\kappa && (3.205)\\
&= u_{\bar{\mu},p'}\dot{u}_{\mu,p} - \dot{u}_{\bar{\mu},p'}u_{\mu,p} + imu_{\bar{\mu},p'}u_{\mu,p} && (3.206)
\end{aligned}$$

$$\begin{aligned}
\bar{\sigma} &\equiv \bar{\alpha} - im\bar{\kappa} && (3.207)\\
&= \sigma(m \to -m) && (3.208)
\end{aligned}$$

Notice that the symmetry between $t_1 \leftrightarrow t_2$, $\mu \leftrightarrow \bar{\mu}$, $p \leftrightarrow p'$ and their conjugate. The sign of mass term switches because γ^0 has eigenvalue $\mp i$ for two upper and two lower components, respectively. Therefore, when we carry out the spin sum, the term with changing $m \to -m$ arises when we take the trace of the matrices. Recall that the mode solutions of Dirac's equations for massive fermion are

$$\begin{aligned}
u_\mu(x) &= \frac{i\sqrt{\pi x}}{2(2\pi)^{\frac{3}{2}}}e^{\frac{\pi r}{2}}H_\mu^{(1)}(x) && (3.209)\\
u_{\bar{\mu}}(x) &= \frac{i\sqrt{\pi x}}{2(2\pi)^{\frac{3}{2}}}e^{\frac{-\pi r}{2}}H_{\bar{\mu}}^{(1)}(x) && (3.210)
\end{aligned}$$

for $\mu = \frac{1}{2} - \frac{im}{H}$, and $\bar{\mu} = \frac{1}{2} + \frac{im}{H}$. The factor $e^{\pm\frac{\pi m}{2H}}$ arises due to the fixing of the coefficients c_p of mode solutions at deep inside horizon with flat space time solutions. We can check that in the massless limit $m = 0$, then $u_\mu = u_{\bar{\mu}} = u_{\frac{1}{2}} = \frac{1}{\sqrt{2}(2\pi)^{\frac{3}{2}}} e^{-ip\tau}$ and $\sigma(t) = \bar{\sigma}(t) = \frac{i}{2(2\pi)^3 a}(p' - p)e^{-i(p+p')\tau}$. Hence,

$$\begin{aligned} \mathcal{M}|_{m=0} &= \frac{4}{a_1^3 a_2^3}\sigma_2^* \sigma_1 (1 + \hat{p}\cdot\hat{p}') && (3.211) \\ &= \frac{1}{(2\pi)^6 a_1^4 a_2^4}(p - p')^2 (1 + \hat{p}\cdot\hat{p}') e^{-i(p+p')(\tau_1 - \tau_2)} && (3.212) \end{aligned}$$

which agrees with the result in massless fermion section in eq. (3.77).

Substituting eqs. (3.81) and (3.82) back in the power spectrum formula eq. (2.46) with $V(t) = -2\epsilon H a^5(t)$, we have $\langle\zeta\zeta\rangle$ due to massive fermion loop as

$$\begin{aligned} \int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')} \Big\langle \zeta(\mathbf{x},t)\zeta(\mathbf{x}',t) \Big\rangle_{loop} &= -\frac{4(2\pi)^3 (8\pi G H^2)^2}{q^6} && (3.213) \\ &\times \int d^3p d^3p' \delta^3(\mathbf{q} + \mathbf{p} + \mathbf{p}')\mathcal{I} && (3.214) \end{aligned}$$

where

$$\mathcal{I} = Re \int_{-\infty}^{\infty} dt_2 a_2^3 (e^{-iq\tau_2} - e^{iq\tau_2}) \int_{-\infty}^{t_2} dt_1 a_1^3 e^{-iq\tau_1} \mathcal{M}_\psi \qquad (3.215)$$

where $\mathcal{M}_\psi$ is the result contributed by the fermionic part in eq. (3.202) and the exponential terms $e^{\pm iq\tau}$ in eq. (3.215) come from the ζ_q parts.

3.9 What if $q \to 0$?

Let us consider what happen if the external momentum $q \to 0$. The delta function in the momentum integral tells us that $|\mathbf{p}'| = |\mathbf{p} + \mathbf{q}|$. This is the conservation of momentum (but not energy) due to the translation invariance of FRW universe. There is an ambiguity on when we can take limit $q \to 0$, before or after the integrations. If we take the external momentum $q \to 0$ and hence $|\mathbf{p}| \simeq |\mathbf{p}'|$ *before* any integrations, the factor $(p - p')$ in eq. (3.211) indicates that $\mathcal{M}_{m=0} = 0$. However, if we take the limit $q \to 0$ *after* all the time and momentum integrals, we have the non-zero result which momentum dependence goes as $q^{-3} \ln q$, similar to the result in the massless fermion section. Nevertheless, the exact expression in eq. (3.202) is more complicated for massive fermion. We now like to see what the result is when $q = 0$ before performing the integrals in this section.

By considering the integrand contributed by the fermionic part $\mathcal{M}_\psi$ in eq. (3.202) only, when $q = 0, p = p'$ $\bar{\sigma}_p \simeq -\sigma_p$ as seen from eqs. (3.205) and (3.207). Hence, at far outside horizon, the integrand contributed by the fermionic part approaches

$$a_1^3 a_2^3 \mathcal{M}_\psi|_{m\neq 0} \to 2(1 - (\hat{p} \cdot \hat{p}'))\sigma_p^*(t_2)\sigma_p(t_1) \tag{3.216}$$

Since the direction of of momentum $\hat{p}$ associated with $u_{\mathbf{p},s}$ is *anti parallel* to the direction of momentum $\hat{p}'$ associated with $v_{\mathbf{p}',r}$ (because of the delta function $\delta^3(\mathbf{p}+\mathbf{p}' + \mathbf{q})$ in the momentum integral), $\mathcal{M}_\psi$ approaches

$$a_1^3 a_2^3 \mathcal{M}_\psi|_{m\neq 0} \to 4\sigma_p^*(t_2)\sigma_p(t_1) \tag{3.217}$$

Notice that the term like $\sigma_p^*(t_2)\sigma_p(t_1)$ is similar to a real scalar field. In particular, if we write a real scalar field in mode function in terms of creation and annihilation operators, the expectation value of a real scalar field in free field vacuum at two different space and time is

$$\langle \varphi(\mathbf{x}_1, t_1)\varphi(\mathbf{x}_2, t_2)\rangle_0 = \int d^3p e^{i\mathbf{p}\cdot(\mathbf{x}_1 - \mathbf{x}_2)} \varphi_p^*(t_2)\varphi_p(t_1) \tag{3.218}$$

This is in fact the tree level of the real scalar field. We normally use this tree two point function at different space and time to construct the in-in propagators. This case is also similar to the condensation of fermion and anti-fermion pairs in the theory of superconductivity and vacuum of QCD. The fermion pairs must have zero total momentum ($\mathbf{p} + \mathbf{p}' = 0$) and angular momentum.

Note that the results above come from the fact that we take the limit $q \to 0$ *before* doing any integrations. We will see the case when we take limit $q \to 0$ *after* some integrations in the next section.

3.10 The Momentum Dependence

So far, the result in eq. (3.202) is exact. There is no approximation involved in eq. (3.202). However, the exact result involves the integrand of Hankel's functions and their time derivatives and this makes the integration quite challenging. Nevertheless, we get some idea that the time integrals will converge and go at most $(\log a)^2$ since we have counted the power of a in the previous section and find that they are of of safe interaction type which goes at most a^0. However, in order to determine the momentum dependence of the power spectrum, we need to integrate over unequal times t_1, t_2 and momentums p, p' associated with fermion fluctuations. We will first integrate over times and then over momentums. The direct calculation is complicated because it involves integrating over products of Hankel's functions with complex order. Since p is the running momentum from 0 to ∞ whereas q is the fixed external momentum associated with a conserved quantity ζ_q, it is helpful to divide the integrals over momentum p in eq. (3.213) as an integral when $\Lambda q \leq p \leq \infty$ and $0 \leq p \leq \Lambda q$. The first integral when $p \to \infty$ can be approximated as if fermion is massless due to its high momentum. So, the result at high momentum after dimensional regularization will be close to that in the massless case in eq. (3.96). The second integral when $p \leq \Lambda q$ indicates that the mass effect may become important in the result. Therefore, additional calculation is needed to determine the momentum dependence.

Since the momentum p corresponds to fermion fluctuation ψ_p and the momentum

q corresponds to the conserved quantity (gravity) ζ_q fluctuation, this implies that massive fermion ψ_p exits the horizon *before* ζ_q exits the horizon if Λ is in the order of 1. In other words, the fermion fluctuation $a^3\bar{\psi}_{p'}\psi_p$ is frozen by the time the fluctuation ζ_q crosses the horizon. At outside horizon of ζ_q, p, p' are sufficiently small and Hankel's function can be approximated as [11]

$$H_\beta^{(1)}(x) \approx -\frac{i\Gamma(\beta)}{\pi}\left(\frac{x}{2}\right)^{-\beta} \tag{3.219}$$

which is valid for $x \ll 1$ and $\beta > 0$. Hence, the mode solutions in eq. (3.209) become

$$u^o_\mu(x) = \frac{\Gamma(\mu)e^{\frac{\pi r}{2}}}{(2\pi)^2}(\frac{x}{2})^{ir} \tag{3.220}$$

$$u^o_{\bar{\mu}}(x) = \frac{\Gamma(\bar{\mu})e^{\frac{-\pi r}{2}}}{(2\pi)^2}(\frac{x}{2})^{-ir} \tag{3.221}$$

where $\mu = \frac{1}{2} - ir$, $\bar{\mu} = \frac{1}{2} + ir$, and $r = \frac{m}{H}$. With this approximation, their (conformal) time derivatives are

$$u'^o_\mu = \frac{ir}{\tau}u^o_\mu, u'^o_{\bar{\mu}} = -\frac{ir}{\tau}u^o_{\bar{\mu}} \tag{3.222}$$

Hence,

$$\sigma^o = -imu^o_{\bar{\mu},p'}u^o_{\mu,p} \tag{3.223}$$

$$u^o_{\bar{\mu},p'}u^o_{\mu,p} = \frac{|\Gamma(\mu)|^2}{(2\pi)^4}p^{ir}p'^{-ir} \tag{3.224}$$

$$\sigma_2^{*o}\sigma_1^o = \bar{\sigma}_2^{*o}\bar{\sigma}_1^o = \frac{m^2|\Gamma(\mu)|^4}{(2\pi)^8} \tag{3.225}$$

$$\sigma_2^{*o}\bar{\sigma}_1^o = (\bar{\sigma}_2^{*o}\sigma_1^o)^* = -\frac{m^2|\Gamma(\mu)|^4}{(2\pi)^8}p'^{2ir}p^{-2ir} \tag{3.226}$$

Notice that $\sigma(p, p')$ and $\bar{\sigma}(p, p')$ are both *time independent*. The exponential factors $e^{\pm\pi r}$ arise when we fix the coefficients to match the solutions with those of inside horizon. Hence, from eq. (3.202)

$$\begin{aligned}\mathcal{M}_\psi^o &= \frac{2m^2|\Gamma(\mu)|^4}{(2\pi)^8 a_1^3 a_2^3}\Big(2-(\hat{p}\cdot\hat{p}')(p'^{2ir}p^{-2ir}+c.c.)\Big) && (3.227)\\ &\equiv a_1^{-3}a_2^{-3}\mathcal{F}^o_{p,p'} && (3.228)\end{aligned}$$

where the factors $a_{1,2}^{-3}$ from the fermionic part will cancel with the factor $\sqrt{-\bar{g}}$ in eq. (3.215). Therefore, $\mathcal{F}^o_{p,p'}$ is constant and hence does not enter in the time integrals.

$$\begin{aligned}\mathcal{I}^o &= \mathcal{F}^o_{p,p'} Re\int_{-\infty}^{\infty} dt_2(e^{-iq\tau_2}-e^{iq\tau_2})\int_{-\infty}^{t_2} dt_1 e^{-iq\tau_1} && (3.229)\\ &= \frac{\mathcal{F}^o_{p,p'}}{H^2(t_q)} Re\int_{-\infty}^{0}\frac{d\tau_2}{\tau_2}(e^{-iq\tau_2}-e^{iq\tau_2})\int_{-\infty}^{\tau_2}\frac{d\tau_1}{\tau_1}e^{-iq\tau_1} && (3.230)\end{aligned}$$

Notice that the time integrals *only* come from the ζ propagators. Because the integrand contributed by the fermionic part $\mathcal{F}^o_{p,p'}$ is time independent so the time integrals can be evaluated independently from the momentum integrals. Although we see from eq. (3.229) that the time integral is in the order of $(\log a)^2$, we need to evaluate this integral if we want to see the momentum dependence q for the correlation function of ζ.

From eq. (3.229), we have

$$\mathcal{I}^o = \frac{\mathcal{F}^o_{p,p'}}{H^2(t_q)} Re\int_{-\infty}^{0}\frac{d\tau_2}{\tau_2}(e^{-iq\tau_2}-e^{iq\tau_2})Ei(-iq\tau_2) \qquad (3.231)$$

Using $Ei(-ix)=ci(x)-isi(x)$, we have

$$\mathcal{I}^o = -\frac{2\mathcal{F}^o_{p,p'}}{H^2(t_q)}\int_{-\infty}^{0}\frac{d\tau_2}{\tau_2}\sin(q\tau_2)si(q\tau_2) \qquad (3.232)$$

With Mathematica,

$$\int_{-\infty}^{0}\frac{d\tau_2}{\tau_2}\sin(q\tau_2)si(q\tau_2) = -\frac{\pi^2}{8} \qquad (3.233)$$

Therefore,

$$\mathcal{I}^o = \frac{\pi^2 \mathcal{F}^o_{p,p'}}{4H^2} \tag{3.234}$$

Notice that the result of the time integral above is q-independent. Now, we can evaluate the momentum integral in eq. (3.213) with the integrand in eqs. (3.227) and (3.234). Hence,

$$\int_{|p-q|}^{p+q} p'dp'\mathcal{F}^o_{p,p'} = G^o(p+q) - G^o(|p-q|) = 2q\frac{\partial G^o}{\partial p} + \mathcal{O}(q^3) \simeq 2pq\mathcal{F}^o_{p,p'=p} \tag{3.235}$$

Note that we only keep the leading order in q in the last equation. The $\hat{p}\cdot\hat{p}' = cos\theta = \frac{p^2-p'^2-q^2}{2pq}$ term in eq. (3.227) becomes $-\frac{q}{2p}$ when eq. (3.235) is used. Hence,

$$\frac{2\pi}{q}\int_0^q dpp\int_{|p-q|}^{|p+q|} dp'p'\mathcal{I}^o = \frac{\pi^3}{H^2}\int_0^q p^2 dp\mathcal{F}^o_p \tag{3.236}$$

$$= \frac{4\pi^3|\Gamma(\mu)|^4 m^2}{(2\pi)^8 H^2}\int_0^q dpp^2(1+\frac{q}{2p}) \tag{3.237}$$

$$= \frac{7\pi^3|\Gamma(\mu)|^4 q^3 m^2}{3(2\pi)^8 H^2} \tag{3.238}$$

Substituting eqs. (3.236) into (3.213), we have the ζ correlation function due to massive fermion loop as

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}\Big\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle_{loop} = -\frac{28\pi^3|\Gamma(\mu)|^4(8\pi G)^2 m^2 H^2}{3(2\pi)^5 q^3} \tag{3.239}$$

Using $|\Gamma(\mu)|^2 = |\Gamma(\frac{1}{2}\pm ir)|^2 = \frac{\pi}{\cosh\pi r}$, hence

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}\Big\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle_{loop} = -\frac{7(8\pi G)^2 m^2 H^2}{24q^3\cosh^2\frac{\pi m}{H}} \tag{3.240}$$

which goes as q^{-3} at low momentum.

The fermion mass m can be arbitrary from very small to as large as M_{Pl}. It should be noted that the power spectrum will *not* be zero when we take $m = 0$. Eq. (3.240) for massive fermion is the result when the fermion pairs $\bar{\psi}_{p'}\psi_p$ exit horizon *before or the same time* as ζ_q exits the horizon ($p \leq q$). We keep only the most dominant mode solutions for massive fermion after horizon exit. For massless fermion case, the solution is simple enough so that we can do the integration exactly without any approximations as done in previous sections. The integrand contributed by massive fermion $\mathcal{F}^o \equiv a_1^3 a_2^3 \mathcal{M}_\psi$ becomes *frozen* after horizon exit (apart from that factor a^{-3} that always arises to cancel the factor $\sqrt{-\bar{g}}$). The negative power of $(-\tau)$ arises in the time integrals $\int\int dt_2 dt_1$ of the massive, but not massless, and fermion is more important than the exponential function when $\tau \to 0$. Therefore, *massive fermion loop can contribute the* $(\log a)^2$ *factor*(for two point function calculated here) because the interaction goes as a^0, whereas $\log a$ does not arise in massless fermion because the interaction goes as a^{-1} rather than a^0.

3.11 Large Coupling

As mentioned in the introduction, we can have fermion coupling to inflaton as large as M_{Pl}. With the large vertex in order of M_{Pl}, it raises the possibility that loop spectrum may not get suppressed by the additional factor of G. However, such large M_{Pl} vertex can only exist in the massive theory. The reason is that inflaton fluctuates around non-zero background and hence it contributes to the second order of action after expansion. Therefore, this requires us to consider not only the vertices, but also the propagators.

We still can use the result as in the previous section even if it has this kind of additional large M_{Pl} coupling. The fermion mass can arise from the non-zero vacuum expectation value of a scalar field in a flat potential. Therefore, the effective fermion mass during inflation can be as large as M_{Pl}. We know that in order to generate all matter observed today, inflaton φ must couple to matter such as fermion some-times during inflation. For example, if fermion is Yukawa coupled to the inflaton $\varphi = \bar{\varphi} + \delta\varphi$, the quantum correction to the ζ correlation function only arises due to the interaction of fermion and gravity fluctuations but not due to the interaction of fermion and fluctuation of scalar field $\delta\varphi$ because we have worked in the $\delta\varphi = 0$ gauge. Therefore, the Yukawa coupling that can arise for general inflaton potential does not change the result in eq. (3.240) but only shift the fermion mass to be

$$m \to m + \bar{\varphi}(t_q) \tag{3.241}$$

However, during inflation fermion mass can be larger than H because the non-zero expectation value of the scalar field $\bar{\varphi}(t)$ is large. We only assume m is constant since $\bar{\varphi}$ does not change very much during slow roll inflation, so we use the value of fermion mass when the inflaton is about at the time of Horizon exit (in the same way as we approximate $H(t) \simeq H(t_q)$ during inflation). Estimating how large the the unperturbed inflaton amplitude can be is perhaps depending on the details of inflaton potential. But we can get some rough estimation via the slow roll condition as

$$\frac{|V'|}{|V|} \ll \sqrt{8\pi G}, \tag{3.242}$$

Therefore, for the coupling of inflaton with matter $V = \bar{\varphi}\bar{\psi}\psi$, the slow roll condition above requires $\bar{\varphi} \gg \frac{1}{\sqrt{8\pi G}}$. Therefore, if unperturbed inflaton amplitude is in the order of M_{Pl} at the time of horizon exit as occurred in many inflationary theories, we have $m \simeq M_{Pl} \equiv \frac{1}{\sqrt{8\pi G}}$ and the power spectrum due to massive fermion loop as

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')} \Big\langle \zeta(\mathbf{x},t)\zeta(\mathbf{x}',t) \Big\rangle_{loop, m=M_{Pl}} = -\frac{7(8\pi G)H^2}{24q^3\cosh^2\frac{\pi M_{Pl}}{H}} \tag{3.243}$$

We see that even when we include the large $\bar{\varphi} \sim M_{Pl}$ coupling that seems to give the quantum effect that does not get suppressed by the factor of G, the result is suppressed by the factor $\cosh^2 \frac{M_{Pl}\pi}{H}$ instead. This happens because inflaton fluctuates around non-zero background, implying that the massive fermion propagators are in the order of M_{Pl}. The factor $\cosh \frac{\pi m}{H}$ arises when we fix the mass dependent coefficients of the mode solution at late time.

It should be mentioned here that the large mass term *does not* get suppressed to the quantity like $a^3\langle\bar{\psi}\psi\rangle \to \tanh \frac{\pi m}{H}$. Terms like $\bar{u}u$ or $\bar{v}v$ approach constant with the mass dependent constant coefficient goes as $\tanh \frac{\pi m}{H}$, which does not have large mass suppression. However, to close the fermion loop, the tree $\langle\bar{\psi}\psi\rangle$ is not the only quantity we need to calculate but rather $\langle\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2\rangle$, which is the trace over the multiplied matrices such as $\sum_{r,s} \bar{v}_{p',r}(t_1)u_{p,s}(t_1)\bar{u}_{p,s}(t_2)v_{p',r}(t_2)$. Therefore the bilinear $\bar{v}_1u_1$ gives constant mass dependent coefficient of $\frac{1}{\cosh \frac{m\pi}{H}}$, resulting in small loop result even when the fermion couples to inflaton in the order of M_{Pl}.

Apart from ultraviolet divergence, no infrared divergence can arise due to the late time behavior. The reason of infrared safe comes from the fact that the function $\mathcal{F}^o$ approaches constant at low momentum. This is similar to viewing σ and $\bar{\sigma}$ in eq. (3.202) as scalar which approach constant after horizon exit. Provided that the integral over time is infrared safe, the integral over time only comes from the ζ correlator, where as the fermionic part only contributes a a^{-3} factor that always cancels with the factor $\sqrt{-\bar{g}}$ in each interaction Hamiltonian. After all integrations, the power spectrum gives a q^{-3} momentum dependence. The result of the massive fermion case is valid at low momentum mode only where we need to cut off the momentum integral p to some value i.e., Λq, so that the approximation of small p, p' in the Hankel function (eq. (3.219)) is still valid. This case means that, if Λ is in the order of 1, the fermion momentum mode p, p' exit the horizon *before* ζ momentum mode q crosses the horizon ($p, p' \leq q = a(t_q)H(t_q)$). For higher momentum mode p, the fermion behaves like it is massless and is always suppressed by the factor G and negative power of Robertson Walker a as shown in massless fermion section. Therefore, the spectrum of massive fermion loop is more dominant than the spectrum of massless fermion loop in general because the former can contribute a $(\ln a)^n$ factor in the result of n time integrals. Bilinear term like $\bar{\psi}\psi$ or $\psi^\dagger\psi$ gives the power of a as a^0 in the interactions. [8] However, $\ln a$ never arise in massless fermion because the interaction Hamiltonian always associates with either its time derivative which goes as $\dot{\psi} \sim a^{-\frac{5}{2}}$[9] or its spatial derivative which is always accompanied by p_i/a. Hence, massless fermion can only give a^{-1} at most.

More careful consideration is needed for the mass effect to investigate whether the quantum effect is truly small. The reason is that various mass dependent coefficients can arise when matching the general solution with that of inside horizon for the general graphs. However, fermion has two components that are needed to form a pair with its conjugate. The bilinear like $\bar{u}u$ or $\bar{v}v$, but *not* $\bar{v}u$, contributes constant factors like $|\Gamma(\nu)|^2(e^{\frac{\pi m}{H}} - e^{-\frac{\pi m}{H}})$ which are $\tanh\frac{\pi m}{H} \to 1$ at large mass limit. The bilinear like $u^\dagger u$ or $v^\dagger v$ contributes constant factor like $\frac{\cosh\frac{m\pi}{H}}{\cosh\frac{m\pi}{H}} = 1$, which is mass independent. Therefore, by considering this alone, fermion has no exponential suppression and seems to give large quantum effect if vertex is as large as M_{Pl}. However, as shown in the detail calculation here, this is not possible for the loop graph that has two external legs with two trilinear vertices because it requires bilinear like $\bar{v}u$ instead. The bilinear term like $\bar{\psi}\gamma^i\psi$ get suppressed at late time because γ^i can only have contraction with $\frac{q_i}{a}$. Therefore it is suppressed by additional negative power of a and its low momentum at outside horizon. The interaction term like $\bar{\psi}\gamma^0\dot{\psi}$ contributes both $\frac{q_i}{a}\bar{\psi}\gamma^i\psi$ and $m\bar{\psi}\psi$ factors via Dirac's equation and its conjugate so it can give the result at most as that in $\bar{\psi}\psi$ interaction type. The other power of bilinear term like $(\bar{\psi}\psi)^n$ for $n > 1$ cannot couple to the mass dimension in the order

[8] Due to the anti-commuting nature of fermion, the dominant mode and dominant mode $\psi \propto a^{-\frac{3}{2}}$ does not cancel in the anti commutator. This is unlike the bosonic cases which satisfy commutation relations and the dominant and dominant mode are cancelled.

[9] As mentioned earlier, the terms proportional to $a^{-\frac{3}{2}}$ in $\dot{\psi}_q$ are cancelled in the gravitational and fermionic interactions so the next leading power of a in $\dot{\psi}_q$ is $a^{-\frac{5}{2}}$

of M_{Pl} and can only give higher fermion loop(by dimension counting in the action). Hence, $(\bar{\psi}\psi)^{n>1}$ interaction type is expected to be suppressed by negative power of $a(t)$.

Chapter 4

Gauge Field

Time is so precious, time is not for sale in the market.

Even for the wealth of three worlds you can't buy back the moment past.

Remembering past moments, don't uselessly be obsessed.

Past wealth can be recovered but past moments can never return.

Living in the past is agitating, living in the future is delusory.

If you live in the present, you have learn how to live. S.N. Goenka

A classical vector field in inflating universe is studied in [12, 13]. We study the *quantum* effect of cosmological density fluctuation due to vector field interacts with gravity in this chapter.

4.1 Gauge Field, Inflaton, and Gravity

We consider the quantized gauge field that affects the quantity ζ and its correlation function through the interaction with gravitational fluctuation. The action is

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_g + \mathcal{L}_\varphi + \mathcal{L}_V && (4.1)\\ &= -\frac{\sqrt{-g}}{2}\Big[\frac{1}{8\pi G}R + g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + 2V && (4.2)\\ &\quad +\frac{1}{2}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} + m^2 g^{\mu\nu}A_\mu A_\nu\Big] && (4.3)\end{aligned}$$

The general action of massive gauge field with gravity is

$$\begin{aligned}\mathcal{L}_V &= -\frac{1}{2}\sqrt{-g}[\frac{1}{2}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} + m^2 g^{\mu\nu}A_\mu A_\nu] && (4.4)\\ &= -\frac{1}{2}\sqrt{-g}\Big[(g^{00}g^{ij} - g^{0i}g^{0j})F_{0i}F_{0j} + 2g^{0i}g^{jk}F_{ji}F_{k0} + \frac{1}{2}g^{ij}g^{lk}F_{li}F_{kj} && (4.5)\\ &\quad +m^2 g^{ij}A_iA_j\Big] && (4.6)\end{aligned}$$

The interaction action of gauge field with gravitational fluctuations can be written in term of ζ and A_i as

$$\begin{aligned}\mathcal{L}_V &= \frac{1}{2}\frac{ae^\zeta}{(1+\frac{\dot\zeta}{H})^2}F_{0i}F_{0i} - \frac{N^i}{(1+\frac{\dot\zeta}{H})^2}\Big(ae^\zeta\delta_{jk} - \frac{a^3e^{3\zeta}N^jN^k}{(1+\frac{\dot\zeta}{H})^2}\Big)F_{ji}F_{k0} && (4.7)\\ &\quad -\frac{1}{4}\Big(ae^\zeta\delta_{ij} - \frac{a^3e^{3\zeta}N^iN^j}{(1+\frac{\dot\zeta}{H})^2}\Big)\Big(a^{-2}e^{-2\zeta}\delta_{lk} - \frac{N^lN^k}{(1+\frac{\dot\zeta}{H})^2}\Big)F_{li}F_{kj} && (4.8)\\ &\quad +m^2\Big(ae^\zeta\delta_{ij} - \frac{a^3e^{3\zeta}N^iN^j}{(1+\frac{\dot\zeta}{H})^2}\Big)A_iA_j + ... && (4.9)\end{aligned}$$

where we choose gauge $A_0 = 0$ for $m = 0$ and $\delta\varphi = 0$ so that the ADM metric N and N^i can be written in term of ζ in the same way as done in chapter 3 in which

$$N^i = -\frac{1}{a^2H}\partial_i\zeta + \epsilon\partial_i\nabla^{-2}\dot\zeta \qquad (4.10)$$

For $m \neq 0$, there are additional terms resulting from solving constraint equation of A_0. They are all subdominant after horizon exit in the order of $\mathcal{O}\Big(\frac{q_i q_j \dot{A}_i \dot{A}_j}{q^2+(am)^2}\Big)$ or smaller, represented by ... in the action above.

4.2 Field Equation and Its Solutions

We expand the gravity and vector field fluctuations as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \tag{4.11}$$

$$A_\mu = 0 + \delta A_\mu \tag{4.12}$$

Hence we have

$$\mathcal{L}_V = \mathcal{L}_V^{(2)} + \mathcal{L}_V^{(3)} + ... \tag{4.13}$$

where $\mathcal{L}_V^{(2)}$ is the second order in fluctuation that gives the field equation and propagator via its solution. $\mathcal{L}_V^{(3)}$ and higher order terms give interaction vertices of vector and gravitational fields fluctuations.

To determine vector field propagator, we need to solve interaction free field equation in inflating universe. It is

$$\partial_\mu\Big(a^3(t)F^{\mu\nu}\Big) - a^3(t)m^2A^\nu = 0 \tag{4.14}$$

For $\nu = 0$, this gives

$$A_0 = -\frac{iq_i\dot{A}_i}{q^2+(ma)^2} \tag{4.15}$$

For $\nu = j$, this gives

$$\ddot{A}_j + H\dot{A}_j + \frac{q^2}{a^2}A_j + m^2 A_j = \frac{q_j q_i A_i}{a^2} + iq_j(\dot{A}_0 + HA_0) \tag{4.16}$$

To eliminate auxiliary field A_0, we apply ∂_ν to eq. (4.14). We have

$$\partial_\nu(a^3 A^\nu) = 0 \tag{4.17}$$

or

$$\dot{A}_0 + 3HA_0 - \frac{iq_i}{a^2}A_i = 0 \tag{4.18}$$

Substituting (4.18) in (4.16), we have dynamical field equation of A_j in expanding universe as

$$\ddot{A}_j + H\Big(1 + \frac{2q_i q_j}{q^2 + (ma)^2}\Big)\dot{A}_j + \Big(\frac{q^2}{a^2} + m^2\Big)A_j = 0 \tag{4.19}$$

For transverse direction $q_i A_i = 0$, we have

$$\ddot{A}_j + H\dot{A}_j + \Big(\frac{q^2}{a^2} + m^2\Big)A_j = 0 \tag{4.20}$$

where this is valid for photon ($m = 0$) and massive vector boson in transverse direction ($\lambda = 1, 2$).

For parallel direction ($\lambda = 3, m \neq 0$ only), we have

$$\ddot{A}_j + \Big(1 + \frac{2q^2}{q^2 + (ma)^2}\Big)\dot{A}_j + \Big(\frac{q^2}{a^2} + m^2\Big)A_j = 0 \tag{4.21}$$

It is impossible to solve equation (4.21) exactly[13]. However, at late time during inflation, $a(t)$ grows more or less exponentially. Therefore, the third term in equation above maybe negligible. Hence, the vector field can be written as

$$A_i(\mathbf{x},t) = \int d^3q \sum_{\lambda} \Big[e^{i\mathbf{q}\cdot\mathbf{x}} e_i(\hat{q},\lambda)\alpha(\mathbf{q},\lambda)\mathcal{A}_q(t) + e^{-i\mathbf{q}\cdot\mathbf{x}} e_i^*(\hat{q},\lambda)\alpha^*(\mathbf{q},\lambda)\mathcal{A}_q^*(t)\Big] \quad (4.22)$$

where

$$\sum_{\lambda,\lambda'=1}^{2} e_i^*(\hat{q},\lambda)e_j(\hat{q},\lambda') = \delta_{ij} - \hat{q}_i\hat{q}_j \quad (4.23)$$

for photon $m = 0$ and

$$\sum_{\lambda,\lambda'=1}^{3} e_i^*(\hat{q},\lambda)e_j(\hat{q},\lambda') \to \delta_{ij} \quad (4.24)$$

for massive vector boson during late time inflation. Therefore, $\mathcal{A}_q(t)$ is the solution that satisfies

$$\frac{d}{dt}\Big(a(t)\frac{d}{dt}\mathcal{A}_q(t)\Big) + \frac{q^2}{a(t)}\mathcal{A}_q(t) + m^2 a\mathcal{A}_q(t) = 0 \quad (4.25)$$

To solve the equation above at a general momentum q, we can work in conformal time τ. Hence the massive gauge field equation in inflating universe is

$$\frac{d^2\mathcal{A}_q}{d\tau^2} + \Big(q^2 + \frac{r^2}{\tau^2}\Big)\mathcal{A}_q = 0 \quad (4.26)$$

where $r \equiv \frac{m}{H}$.

We see from eq. (4.26) that in the limit of $m = 0$, the solution of massless vector field is a plane wave, which is the same as those of conformal scalar and massless fermion[1], due to the conformal flatness of the theories. The positive mode solution of massless vector field at general wavelength is

$$\mathcal{A}_q(t) = \frac{1}{(2\pi)^{\frac{3}{2}}\sqrt{2q}} e^{-iq\tau}, m = 0 \quad (4.27)$$

[1]Apart from spinor and polarization factors

For massive vector field $m \neq 0$, the field equation (4.26) has the Bessel's equation type[11]

$$u_q'' + \Big(q^2 - \frac{4\nu^2 - 1}{4\tau^2}\Big)u_q = 0 \tag{4.28}$$

Therefore, the general solution of massive vector field is

$$\mathcal{A}_q(\tau) = \mathcal{E}_q\sqrt{-\tau}H_\nu^{(1)}(-q\tau) + \mathcal{F}_q\sqrt{-\tau}H_\nu^{(2)}(-q\tau), m \neq 0 \tag{4.29}$$

where

$$\nu^2 = \frac{1}{4} - r^2 \tag{4.30}$$

Therefore, ν is complex for very heavy mass when $m > \frac{H}{2}$. Since we want the solution to match with the positive solution at deep inside horizon $e^{-i\omega\tau}$, only $H_\nu^{(1)}(x)$ but not $H_\nu^{(2)}(x)$ gives $e^{-i\omega\tau}$ factor in the large $|x|$ limit. Hence, $\mathcal{F}_q = 0$ and

$$\mathcal{A}_q(t) \quad = \quad \mathcal{E}_q\sqrt{-\tau}H_\nu^{(1)}(-q\tau) \tag{4.31}$$

A normalized constant $\mathcal{E}_q$ is chosen to match with solution at deep inside horizon. Inside horizon, the positive frequency solution is the same as that in flat space, which is,

$$\mathcal{A}_q(t) \quad \rightarrow \quad \frac{1}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_q}} \exp\Big(-i\int_{-\infty}^{\tau}\omega_q(\tau')d\tau'\Big) \tag{4.32}$$

$$\tag{4.33}$$

where $\omega_q(\tau) \equiv \sqrt{q^2 + (ma)^2}$. With the property of Hankel's function in asymptotic limit,$|x| \rightarrow \infty$

$$H_\nu^{(1)}(x) \rightarrow \sqrt{\frac{2}{\pi x}} \exp\Big(i(x - \frac{\pi\nu}{2} - \frac{\pi}{4})\Big) \tag{4.34}$$

Since we now allow the existence of mass term which can be either large or small, the normalized constants $\mathcal{E}_q$ can be a function of mass and this may affect the result of correlation function.

During inflation, the mass of the matter can be large due to the interaction of matter with inflaton $\bar{\varphi}$. As mentioned earlier, slow roll condition of some inflationary theories requires $m = \bar{\varphi} \simeq M_{Pl}$. This can make the mass term as large as M_{Pl} and may affect the final result of the correlation function. To determine the time independent coefficient $\mathcal{E}_q$, we match the solution with that of inside horizon. From eqs. (4.31), (4.32), and (4.34), we have the mass dependent coefficient $\mathcal{E}_q$ as

$$\mathcal{E}_q(m) = \frac{\sqrt{\pi}}{2(2\pi)^{\frac{3}{2}}} e^{\frac{i\pi}{4}(1+2\nu)} \tag{4.35}$$

From eqs. (4.31) and (6.31), we therefore have the massive mode solution of gauge field that we will use for the propagator as

$$\mathcal{A}_q(t) = \frac{\sqrt{\pi}}{2(2\pi)^{\frac{3}{2}}} e^{\frac{i\pi}{4}(1+2\nu)} \sqrt{-\tau} H_\nu^{(1)}(-q\tau) \tag{4.36}$$

where $\nu = \sqrt{\frac{1}{2} - r^2}$. We see that the gauge field has many different features similar to that of scalar field. The difference between gauge field and scalar field is that we need to consider the polarization vector e_i as well.

4.3 Gauge and Gravity Interaction Vertices

The action $\sqrt{-g} g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} F_{\mu\nu}$ gives the interaction of gauge field with gravity. Expansion of the metric fluctuation gives an infinite number of vertex interactions such as $\mathcal{L}_{\zeta A_i A_j}$, $\mathcal{L}_{\zeta\zeta A_i A_j}$, $\mathcal{L}_{\zeta\zeta\zeta A_i A_j}$,...etc. The field strength $F^{\mu\nu}$ in the action indicates that the interaction always depend on the time or spatial derivative of vector A^i but never depend on the time derivative of A_0 ($F_{00} = 0$ always). For example,

$$F_{ij} = \partial_i A_j - \partial_j A_i \tag{4.37}$$

$$F_{0j} = \dot{A}_j - \partial_j A_0 = \dot{A}_j - \partial_j A_0 \tag{4.38}$$

From the full action in eq.(4.7), the interaction action to cubic order is

$$\mathcal{L}_{\zeta A_i A_i} = \frac{a}{2}\Big(\zeta - \frac{2\dot{\zeta}}{H}\Big)\dot{A}_i^2 + a\Big(-\frac{1}{a^2 H}\partial_i\zeta + \epsilon\partial_i\nabla^{-2}\dot{\zeta}\Big)\Big(\partial_j A_i - \partial_i A_j\Big)\dot{A}_j \tag{4.39}$$

$$+\frac{\zeta}{4a}\Big(\partial_j A_i - \partial_i A_j\Big)^2 + m^2 a\zeta A_i^2 + \mathcal{O}\Big(\frac{q_i q_j \dot{A}_i \dot{A}_j F[\zeta]}{q^2 + (am)^2}\Big) \tag{4.40}$$

We see that the interaction $\mathcal{L}_{\zeta A_i A_j}$ is rather complicated especially when we take into account quantum theory. As learned in appendix, we only need to know the energy momentum tensor of that matter considered to second order because the interaction vertices of any matter can be written as

$$H_{\zeta AA}(t) = -\int d^3x \epsilon H a^5 (T^{00} + a^2 T^{ii})\partial^{-2}\dot{\zeta} + \dot{Y}(t) \tag{4.41}$$

where

$$Y(t) = a^6 T^{00}\big(\frac{\zeta}{Ha^3} - \frac{\epsilon}{a}\partial^{-2}\dot{\zeta}\big) \tag{4.42}$$

This can be verified by using Bianchi identity $T^{\mu\nu}_{;\nu} = 0$ and Mukhanov equation in eq. (3.29). Therefore, we will use eq. (4.41) in loop calculation, instead of eq. (4.39). For electromagnetic field, the energy momentum tensor in the presence of gravity is [7]

$$T^{\mu\nu}_{EM} = F^{\mu}_{\lambda}F^{\nu\lambda} - \frac{1}{4}g^{\mu\nu}F_{\lambda\kappa}F^{\lambda\kappa} \tag{4.43}$$

for $m = 0$. Therefore, to quadratic order of energy momentum tensor, we have

$$(T^{00} + a^2 T^{ii})_{EM} = \frac{1}{a^2}\dot{A}_i^2 + \frac{1}{2a^4}\Big(\partial_i A_j - \partial_j A_i\Big)^2 \tag{4.44}$$

for $A_0 = 0$. Substituting eq. (4.44) in eqs. (4.41), we arrive at

$$H^{EM}_{\zeta AA}(t) = -\int d^3x \epsilon H\Big(a^3 \dot{A}_i^2 + \frac{a}{2}(\partial_i A_j - \partial_j A_i)^2\Big)\nabla^{-2}\dot{\zeta} \tag{4.45}$$

Let us calculate what the interaction trilinear vertices of massive vector field is. We can easily extend the result to the massive case. Since this includes the possibility of gauge field potential such as the mass term i.e. $V(A_\mu A^\mu) = \frac{1}{2}m^2 g^{\mu\nu} A_\mu A_\nu$, the energy momentum tensor is added from the massless case to be

$$T^{\mu\nu}_{m\neq 0} \equiv \frac{2}{\sqrt{-g}}\frac{\delta \mathcal{L}_v}{\delta g_{\mu\nu}} \tag{4.46}$$

$$= F^\mu_\lambda F^{\nu\lambda} - \frac{1}{4}g^{\mu\nu}F_{\lambda\kappa}F^{\lambda\kappa} + m^2\Big(A^\mu A^\nu - \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}A_\alpha A_\beta\Big) \tag{4.47}$$

Therefore, to quadratic order of energy momentum tensor when $A_0 \neq 0$, we have

$$T^{00} + a^2 T^{ii} = \frac{1}{a^2}F_{0i}^2 + \frac{1}{2a^4}F_{ij}^2 + 2m^2 A_0^2 \tag{4.48}$$

As seen from eq. (4.15), A_0 is decaying away as $\frac{q_i \dot{A}_i}{(am)^2}$ after horizon exit. Therefore, we can approximate massive vertices as that of massless case, which is,

$$H_{\zeta AA}(t) = -\int d^3x \epsilon H a^5 \Big(\frac{1}{a^2}\dot{A}_i^2 + \frac{1}{2a^4}(\partial_i A_j - \partial_j A_i)^2\Big)\nabla^{-2}\dot{\zeta} \tag{4.49}$$

Therefore, it is only the propagators that will be different from massless case as we will consider in the next section.

4.4 Massless Vector Field

We would like to determine the momentum dependence of vector field loop spectrum $\langle\zeta\zeta\rangle \sim q^{-n}$ to see whether n is far different from 3 or not. If n is much larger than 3, the spectrum will therefore be larger than the classical result at outside horizon of ζ_q when $q \to 0$ and therefore vector field will not be a large scale structure candidate. Therefore, we would like to calculate the momentum dependence of loop spectrum to see whether the scale invariance is broken or not.

4.4.1 The Momentum Dependence Result

We can use the general formula in eq.(2.46) with the replacement

$$\psi^*\psi \;\rightarrow\; \frac{1}{a^2}\dot{A}_i^2+\frac{1}{a^4}\Big((\partial_i A_j)^2-\partial_i A_j\partial_j A_i\Big) \tag{4.50}$$

$$\zeta_q(t_{1,2}) \;\rightarrow\; -\dot{\zeta}_q(t_{1,2})/q^2 \tag{4.51}$$

$$V(t) \;=\; -\epsilon H a^5 \tag{4.52}$$

As seen above, it is straight forward to extend from the scalar case to gauge field. The only difference is that we need to consider the polarization vector e_i. Since the purely electric term(through $(\dot{A}_i)^2$), purely magnetic term (through $(\partial_i A_j)^2 - (\partial_i A_j)(\partial_j A_i)$), and two cross terms arise when we evaluate commutator $[H_1,[H_2,Q]]$. Therefore, eq. (2.53) becomes

$$\mathcal{M}_{\mathcal{A}} \;=\; \frac{\mathcal{P}_1}{a_1^2a_2^2}\dot{\mathcal{A}}_p(t_1)\dot{\mathcal{A}}_{p'}(t_1)\dot{\mathcal{A}}^*_p(t_2)\dot{\mathcal{A}}^*_{p'}(t_2)+ \tag{4.53}$$

$$\frac{\mathcal{P}_2}{a_1^4a_2^4}\mathcal{A}_p(t_1)\mathcal{A}_{p'}(t_1)\mathcal{A}^*_p(t_2)\mathcal{A}^*_{p'}(t_2)+ \tag{4.54}$$

$$\frac{\mathcal{P}_3}{a_1^4a_2^2}\mathcal{A}_p(t_1)\mathcal{A}_{p'}(t_1)\dot{\mathcal{A}}^*_p(t_2)\dot{\mathcal{A}}^*_{p'}(t_2)+ \tag{4.55}$$

$$\frac{\mathcal{P}_4}{a_2^3a_4}\dot{\mathcal{A}}_p(t_1)\dot{\mathcal{A}}_{p'}(t_1)\mathcal{A}^*_p(t_2)\mathcal{A}^*_{p'}(t_2) \tag{4.56}$$

and ζ remains the same as in eq. (3.72) because we are calculating the same correlation function $\langle\zeta\zeta\rangle$ with various kinds of matter loops.

We now need to calculate the polarization factor $\mathcal{P}_i$ for $i = 1...4$.

The $\mathcal{P}_1$ factor comes from the purely electric field term which is

$$\mathcal{P}_1 \;=\; \sum_{\lambda,\lambda'=1}^{2}(e_{i,p,\lambda}e^*_{j,p,\lambda})(e_{i,p',\lambda'}e^*_{j,p',\lambda'}) \tag{4.57}$$

$$=\; 1+\frac{(\mathbf{p}\cdot\mathbf{p}')^2}{|\mathbf{p}|^2|\mathbf{p}'|^2} \tag{4.58}$$

The $\mathcal{P}_2$ factor comes from the purely magnetic field term which is

$$\mathcal{P}_2 = \sum_{\lambda,\lambda'=1}^{2} |(\mathbf{p}\cdot\mathbf{p'})(\hat{e}_{p,\lambda}\cdot\hat{e}_{p',\lambda'}) - (\mathbf{p}\cdot\hat{e}_{p',\lambda'})(\mathbf{p'}\cdot\hat{e}_{p,\lambda})|^2 \tag{4.59}$$

$$= |\mathbf{p}|^2|\mathbf{p}'|^2 + (\mathbf{p}\cdot\mathbf{p}')^2 \tag{4.60}$$

The $\mathcal{P}_{3,4}$ factors come from the cross terms which are

$$\mathcal{P}_3 = \mathcal{P}_4 = -2\mathbf{p}\cdot\mathbf{p}' \tag{4.61}$$

Substitute these in eq. (2.46), we have

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x'})}\Big\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle_{loop} = -8(2\pi)^9\int d^3pd^3p'\delta^3(\mathbf{q}+\mathbf{p}+\mathbf{p}') \tag{4.62}$$

$$\int_{-\infty}^{t} dt_2\epsilon_2H_2a_2^5\int_{-\infty}^{t_2} dt_1\epsilon_1H_1a_1^5 Re\Big(\mathcal{Z}\mathcal{M}_{\mathcal{A}}\Big) \tag{4.63}$$

where

$$\mathcal{M}_A = \Big(1+(\hat{p}\cdot\hat{p}')^2\Big)a_1^{-2}a_2^{-2}\dot{\mathcal{A}}_p(t_1)\dot{\mathcal{A}}_{p'}(t_1)\dot{\mathcal{A}}_p^*(t_2)\dot{\mathcal{A}}_{p'}^*(t_2) \tag{4.64}$$

$$+p^2p'^2\Big(1+(\hat{p}\cdot\hat{p}')^2\Big)a_1^{-4}a_2^{-4}\mathcal{A}_p(t_1)\mathcal{A}_{p'}(t_1)\mathcal{A}_p^*(t_2)\mathcal{A}_{p'}^*(t_2) \tag{4.65}$$

$$-2pp'(\hat{p}\cdot\hat{p}')a_1^{-4}a_2^{-2}\mathcal{A}_p(t_1)\mathcal{A}_{p'}(t_1)\dot{\mathcal{A}}_p^*(t_2)\dot{\mathcal{A}}_{p'}^*(t_2) \tag{4.66}$$

$$-2pp'(\hat{p}\cdot\hat{p}')a_1^{-2}a_2^{-4}\dot{\mathcal{A}}_p(t_1)\dot{\mathcal{A}}_{p'}(t_1)\mathcal{A}_p^*(t_2)\mathcal{A}_{p'}^*(t_2) \tag{4.67}$$

Since the solution of massless vector field is just a plane wave,

$$\dot{\mathcal{A}}_q(t) = -\frac{iq}{a(t)}\mathcal{A}_q(t) \tag{4.68}$$

Eq. (4.64) can be simplified further as

$$\mathcal{M}_{\mathcal{A}} = \frac{2p^2p'^2}{a_1^4a_2^4}\Big(1+(\hat{p}\cdot\hat{p}')\Big)^2\mathcal{A}_p(t_1)\mathcal{A}_{p'}(t_1)\mathcal{A}^*_p(t_2)\mathcal{A}^*_{p'}(t_2) \tag{4.69}$$

$$= \frac{p^2p'^2}{2(2\pi)^6a_1^4a_2^4pp'}\Big(1+(\hat{p}\cdot\hat{p}')\Big)^2e^{-i(p+p')(\tau_1-\tau_2)} \tag{4.70}$$

Substituting the $\mathcal{Z}$ part eq. (3.82) and gauge field part eq. (4.69) in eq. (4.62), we have the correlation function due to massless vector field as

$$\int d^3xe^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x'})}\Big\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle_{loop} = -16(2\pi)^9\epsilon^2|\zeta_q^o|^4\int d^3pd^3p' \tag{4.71}$$

$$\times\delta^3(\mathbf{q}+\mathbf{p}+\mathbf{p}')|\mathcal{A}_p^o|^2|\mathcal{A}_{p'}^o|^2p^2p'^2\Big(1+(\hat{p}\cdot\hat{p}')\Big)^2\mathcal{T} \tag{4.72}$$

where

$$\mathcal{T} = Re\int_{-\infty}^{0}d\tau_2(e^{-iq\tau_2}-e^{iq\tau_2})e^{i(p+p')\tau_2}\int_{-\infty}^{\tau_2}d\tau_1e^{-i(q+p+p')\tau_1} \tag{4.73}$$

$$= -\frac{1}{2q(q+p+p')} \tag{4.74}$$

and the constant coefficients after horizon exit are

$$|\mathcal{A}_q^o|^2 = \frac{1}{2(2\pi)^3q} \tag{4.75}$$

$$|\zeta_q^o|^2 = \frac{8\pi GH^2}{2(2\pi)^3\epsilon q^3} \tag{4.76}$$

Although the momentum dependence of $|\mathcal{A}_q^o|^2$ goes as q^{-1} rather than q^{-3}, this does not break the scale invariance of the power spectrum $\langle\zeta\zeta\rangle$ because we do not measure the product of the vector fields. Therefore, we study how vector field affects the $\langle\zeta\zeta\rangle$ via the gravity interaction instead. Substituting eqs. (4.73) and (4.75) in

(4.71), we have the loop power spectrum due to massless vector field as

$$\int d^3xe^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x'})}\Big\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle_{loop} = \frac{(8\pi GH^2)^2}{2(2\pi)^3q^7}\int d^3pd^3p' \tag{4.77}$$

$$\times\delta^3(\mathbf{q}+\mathbf{p}+\mathbf{p}')\frac{pp'}{q+p+p'}\Big(1+(\hat{p}\cdot\hat{p}')\Big)^2 \tag{4.78}$$

Notice that the first term is the same as that of massless minimal coupled scalar loop. The $\hat{p}\cdot\hat{p}'$ terms are the consequence of summing over polarization vectors. We see from the momentum integral above that there is the *quartic* UV divergence when $p\to\infty$. We will use dimensional regularization to remove the divergence. Note that eq. (4.77) can be written as

$$\int d^3xe^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}\Big\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle_{loop} = \frac{(8\pi GH^2)^2}{2(2\pi)^3q^7}\Big[\frac{2\pi}{q}\mathcal{I}(q)\Big] \tag{4.79}$$

where

$$\mathcal{I}(q)\equiv\int_0^\infty pdp\int_{|p-q|}^{|p+q|}p'dp'\frac{pp'}{q+p+p'}\Big(1+\frac{p'^2-p^2-q^2}{2pq}\Big)^2 \tag{4.80}$$

With the dimensional regularization, the UV divergence of the integral above for $\delta=0$ gives the pole term as

$$\frac{2\pi}{q}\mathcal{I}(q)\Rightarrow q^{4+\delta}F(\delta) \tag{4.81}$$

$$F(\delta)\to\frac{F_0}{\delta}+F_1 \tag{4.82}$$

Therefore, in the limit $\delta=0$,

$$\frac{2\pi}{q}\mathcal{I}(q)=q^4\Big[F_0\ln q+L\Big] \tag{4.83}$$

where L is a divergent constant. To eliminate the divergence in the momentum integral above, we use Mathematica to integrate over p' and differentiate $\mathcal{I}(q)$ six times to get finite integral over p. Hence,

$$\frac{d^{(6)}\mathcal{I}(q)}{dq^6} = -\frac{336}{q} \tag{4.84}$$

where we take the limit $q \to 0$ *after* integrating over p. Hence,

$$\mathcal{I}(q) = q^5\Big(-\frac{336}{5!}\ln q + L\Big) \tag{4.85}$$

or

$$F_0 = -\frac{28\pi}{5} \tag{4.86}$$

Substituting $\mathcal{I}(q)$ back into eq. (4.79), we have the finite part of correlation function as

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}\Big\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle_{loop} = -\frac{14\pi(8\pi GH^2(t_q))^2}{5(2\pi)^3q^3}\Big(\ln q + L\Big) \tag{4.87}$$

Notice that the result is smaller than the classical result by a factor of G in order of magnitude. The numerical coefficient is slightly more than that of scalar loop in [2] because there are additional polarization factors.

4.5 Massive Vector Field

The actual calculation is more complicated for massive vector field because mode solutions for the propagator is no longer a simple plane wave as in the massless case. We therefore need to study the late time behavior before performing actual integrals.

4.5.1 Late Time Behavior

We see from the full action in eq. (4.7) that none of them has positive power of a as long as the vector field is massless. The reason is that N^i goes as a^{-2} and $\dot{A}_i$ contained in F_{0i} goes as a^{-1} when $m = 0$. Therefore, there is no problem with the time integral in the massless case. However, if vector field has mass, as it is for a more general case[2], the power of a will go explicitly as a in the interaction actions. We therefore consider the late time mode solution of massive vector field.

For $|x| \to 0$,

$$J_\nu(x) \to \frac{x^\nu}{2^\nu \Gamma(\nu+1)}(1 + \mathcal{O}(x^2)) \tag{4.88}$$

By definition of Hankel's function,

$$H_\nu^{(1)} = \frac{1}{i \sin \nu\pi}\Big(J_{-\nu} - e^{-\nu\pi i} J_\nu\Big) \tag{4.89}$$

Hence, the late time behavior of mode solution approaches

$$H_\nu^{(1)} \to -\frac{i}{\sin \nu\pi}\Big(\frac{x^{-\nu}}{2^{-\nu}\Gamma(1-\nu)} - \frac{e^{-\nu\pi i} x^\nu}{2^\nu \Gamma(\nu+1)}\Big)\Big(1 + \mathcal{O}(x)^2\Big) \tag{4.90}$$

The exact solution in eq. (4.36) approaches

$$\mathcal{A}_q(t) = \mathcal{C}_q a^{\lambda_+} + \mathcal{D}_q a^{\lambda_-} \tag{4.91}$$

at late time where

$$\lambda_\pm = -\frac{1}{2} \pm \nu = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{m^2}{H^2}} \tag{4.92}$$

[2]This mass m is an effective mass which includes possible interactions with broken symmetry of other fields such as inflaton. Although the gauge invariant is broken in the action, the mass term always arises because inflaton φ and gravity $g_{\mu\nu}$ fluctuate around non-zero background. After expansion, there is always contribution from at least the second order of A_i which does not involve field derivatives of A_i. Therefore, this contributes an effective mass term of $m^2 \sim \bar{\varphi}^2(t_q) \sim M_{Pl}^2$. For example, if the vector field A_i couples to the inflaton $\bar{\varphi}$, $\bar{\varphi}$ serves as the effective mass in the order of M_{Pl}

and

$$\mathcal{C}_q = -\frac{i\sqrt{\pi}}{2(2\pi)^{\frac{3}{2}}\sqrt{H}\sin\nu\pi}\frac{e^{\frac{i\pi}{4}(1+2\nu)}}{\Gamma(1-\nu)}\Big(\frac{2H}{q}\Big)^{\nu} \tag{4.93}$$

$$\mathcal{D}_q = \frac{i\sqrt{\pi}}{2(2\pi)^{\frac{3}{2}}\sqrt{H}\sin\nu\pi}\frac{e^{\frac{i\pi}{4}(1-2\nu)}}{\Gamma(1+\nu)}\Big(\frac{2H}{q}\Big)^{-\nu} \tag{4.94}$$

Notice that its time derivative contribute the same power of a as

$$\dot{\mathcal{A}}_q(t) = H\Big(\lambda_+\mathcal{C}_q a^{\lambda_+} + \lambda_-\mathcal{D}_q a^{\lambda_-}\Big) \tag{4.95}$$

For $m > \frac{H}{2}$, ν is complex. Therefore the two mode solutions at late time are complex conjugate of each other.

$$\lambda_+ \equiv \lambda, \lambda_- = \lambda^* \tag{4.96}$$

Notice that $|\mathcal{C}_q|^2$ and $|\mathcal{D}_q|^2$ are q-*independent*.
For $m = \frac{H}{2}$, ν is zero. Therefore,

$$\lambda_+ = \lambda_- = -\frac{1}{2} \tag{4.97}$$

and $\mathcal{C}_q$ and $\mathcal{D}_q$ are q- *independent*.
For $m < \frac{H}{2}$, ν is real. Therefore,

$$-\frac{1}{2} < \lambda_+ < 0, -1 < \lambda_- < -\frac{1}{2} \tag{4.98}$$

and $|\mathcal{C}_q|^2$ and $|\mathcal{D}_q|^2$ are q-*dependent*, depending on its mass.
We see from the interaction action that there is at most explicit factor of a^1. If vector field is massless, this causes no problem to the time integrals because its time derivative gives an additional factor of a^{-1}. However, when vector field has mass, its time derivative gives the same power of $\mathcal{A}_q$ at late time. But the commutators

of massive vector field contribute a factor of $a^{\lambda_+ + \lambda_-} = a^{-1}$. As seen from the full action in eq. (4.7), the maximum number of power of a in the action when $m \neq 0$ is 1 and the commutator goes as a^{-1}. Therefore, it can only contribute at most a power of $(\ln a)^N$ in the result of the N time integrals.

Note that the massive theories may provide some interesting features to the final result of power spectrum in that they may not get suppressed by an additional factor of G. Even though the trilinear vertices are the same as those in the massless case (as seen from the vertices section), the time derivative of propagator in massive theories contributes an additional factor

$$\dot{A}_q \to \lambda_\pm f(t) \to \Big(-\frac{1}{2} \pm \sqrt{\frac{1}{2} - \frac{m^2}{H^2}} \Big) f(t) \tag{4.99}$$

where $f(t)$ has the same power of t as in $\mathcal{A}_q$. Therefore, there will be a term like

$$\dot{\mathcal{A}}_p(t_1)\dot{\mathcal{A}}_{p'}(t_1)\dot{\mathcal{A}}^*_p(t_2)\dot{\mathcal{A}}^*_{p'}(t_2) \to \mathcal{O}(\lambda^4_\pm) F[t_1, t_2] \tag{4.100}$$

Since at large mass limit,

$$\lambda^4_\pm \to \frac{M^4_{Pl}}{H^4} \tag{4.101}$$

the time derivative propagators give an additional factor of $(\lambda_\pm)^4$ for four fields. Because of eq. (4.101), the loop spectrum does not seem to get suppressed by an additional factor of G, but however may get suppressed by the constant coefficient at large mass in eqs. (4.93) and (4.94) or the results of the loop integrals. We will investigate whether there is true suppression or not in the next sections.

4.5.2 The Momentum Dependence Result

As seen from the section on vertices, the interaction vertices are the same as those of massless case because of the mass term cancellation in $T^{00} + a^2 T^{ii}$. Only the propagators are different in the massless case. Therefore,

$$\mathcal{M}_{\mathcal{A},m\neq 0} = \Pi^*(t_2)\Pi(t_1) \tag{4.102}$$

where

$$\Pi(t) = \frac{1}{a^2(t)}\dot{A}_{i,p,\lambda}(t)\dot{A}_{i,p',\lambda'}(t) + \frac{1}{a^4(t)}\Big(p_i p'_j A_{j,p,\lambda}(t)A_{i,p',\lambda'}(t) \tag{4.103}$$
$$-p_i p'_i A_{j,p,\lambda}(t)A_{j,p',\lambda'}(t)\Big) \tag{4.104}$$

The exact solution of massive vector field involves Hankel's functions which are rather complicated to integrate over time and momentums. However, we can get some ideas what the momentum dependence of the observable spectrum is by considering the long wavelength mode solutions. Note that at far outside horizon when $q \ll aH, am$, the polarization of massive vector fields approaches

$$\sum_{\lambda=1}^{3} e_i^* e_j \to \delta_{ij} \tag{4.105}$$

Therefore,

$$\Pi^*(t_2)\Pi(t_1) \to \frac{3}{a_2^2 a_1^2}\dot{\mathcal{A}}_p{}^*(t_2)\dot{\mathcal{A}}_{p'}{}^*(t_2)\dot{\mathcal{A}}_p(t_1)\dot{\mathcal{A}}_{p'}(t_1) \tag{4.106}$$
$$+\frac{1}{a_2^4 a_1^4}p^2 p'^2(1+\hat{p}\cdot\hat{p}')\mathcal{A}_p^*(t_2)\mathcal{A}_{p'}^*(t_2)\mathcal{A}_p(t_1)\mathcal{A}_{p'}(t_1) \tag{4.107}$$
$$-\frac{2}{a_1^2 a_2^4}(\mathbf{p}\cdot\mathbf{p}')\mathcal{A}_p^*(t_2)\mathcal{A}_{p'}^*(t_2)\dot{\mathcal{A}}_p(t_1)\dot{\mathcal{A}}_{p'}(t_1) \tag{4.108}$$
$$-\frac{2}{a_1^4 a_2^2}(\mathbf{p}\cdot\mathbf{p}')\dot{\mathcal{A}}_p^*(t_2)\dot{\mathcal{A}}_{p'}^*(t_2)\mathcal{A}_p(t_1)\mathcal{A}_{p'}(t_1) \tag{4.109}$$

We see from eqs. (4.91), and (4.95) that $\dot{\mathcal{A}}_q$ gives the same power of a as $\mathcal{A}_q$ at late time. Therefore, we can keep the most leading order term as the universe rapidly expands as

$$\mathcal{M}_A \quad \rightarrow \quad \frac{3}{a_2^2 a_1^2} \dot{\mathcal{A}}_p{}^*(t_2) \dot{\mathcal{A}}_{p'}{}^*(t_2) \dot{\mathcal{A}}_p(t_1) \dot{\mathcal{A}}_{p'}(t_1) \tag{4.110}$$

This means that the (massive) electric-like term is more dominating than the magnetic-like term after horizon exit. This result is different from the result of massless vector fields in which all electric and magnetic terms are equally important. Substituting eq. (4.110) into eq. (2.46), we have

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')} \Big\langle \zeta(\mathbf{x},t)\zeta(\mathbf{x}',t) \Big\rangle_{loop} \rightarrow -24(2\pi)^9 \int d^3p d^3p' \delta^3(\mathbf{q}+\mathbf{p}+\mathbf{p}') \tag{4.111}$$

$$\int_{-\infty}^{t} dt_2 \epsilon_2 H_2 a_2^3 \int_{-\infty}^{t_2} dt_1 \epsilon_1 H_1 a_1^3 Re\Big(\mathcal{Z} \dot{\mathcal{A}}_p{}^*(t_2) \dot{\mathcal{A}}_{p'}{}^*(t_2) \dot{\mathcal{A}}_p(t_1) \dot{\mathcal{A}}_{p'}(t_1) \Big) \tag{4.112}$$

where $\mathcal{Z}$ is the contribution from the ζ part which is still the same as that in eq. (3.82)

$$\mathcal{Z} = \frac{|\zeta_q^o|^4}{H_1 H_2 a_1^2 a_2^2} \Big(e^{2iq\tau - iq(\tau_1+\tau_2)} - e^{iq(\tau_2-\tau_1)} \Big) \tag{4.113}$$

Hence,

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')} \Big\langle \zeta(\mathbf{x},t)\zeta(\mathbf{x}',t) \Big\rangle_{loop} \rightarrow \tag{4.114}$$

$$-24(2\pi)^9 |\epsilon|^2 |\zeta_q^o|^4 \int d^3p d^3p' \delta^3(\mathbf{q}+\mathbf{p}+\mathbf{p}') \mathcal{T} \tag{4.115}$$

where

$$\mathcal{T} \quad \equiv \quad Re \int_{-\infty}^{t} dt_2 a_2 (e^{-iq\tau_2} - e^{iq\tau_2}) \dot{\mathcal{A}}_p{}^*(t_2) \dot{\mathcal{A}}_{p'}{}^*(t_2) \tag{4.116}$$

$$\times \int_{-\infty}^{t_2} dt_1 a_1 e^{-iq\tau_1} \dot{\mathcal{A}}_p(t_1) \dot{\mathcal{A}}_{p'}(t_1) \tag{4.117}$$

With the time derivative of late time mode solution in eq. (4.95), we have

$$\dot{\mathcal{A}}_p(t_1)\dot{\mathcal{A}}_{p'}(t_1) = H^2\Big[\lambda_+^2\mathcal{C}_p\mathcal{C}_{p'}a_1^{2\lambda_+} + \lambda_-^2\mathcal{D}_p\mathcal{D}_{p'}a_1^{2\lambda_-} \tag{4.118}$$

$$+\lambda_+\lambda_-(\mathcal{C}_p\mathcal{D}_{p'} + \mathcal{D}_p\mathcal{C}_{p'})a_1^{\lambda_++\lambda_-}\Big] \tag{4.119}$$

Therefore, the t_1 integral is

$$\int_{-\infty}^{t_2} dt_1 a_1 e^{-iq\tau_1}\dot{\mathcal{A}}_p(t_1)\dot{\mathcal{A}}_{p'}(t_1) = H^2\int_{-\infty}^{t_2} dt_1 e^{-iq\tau_1}\Big[\lambda_+^2\mathcal{C}_p\mathcal{C}_{p'}a_1^{2\nu} \tag{4.120}$$

$$+\lambda_-^2\mathcal{D}_p\mathcal{D}_{p'}a_1^{-2\nu} + \lambda_+\lambda_-(\mathcal{C}_p\mathcal{D}_{p'} + \mathcal{D}_p\mathcal{C}_{p'})\Big] \tag{4.121}$$

$$\to H\Big[\frac{\lambda_+^2}{2\nu}c_+a_2^{2\nu} - \frac{\lambda_-^2}{2\nu}c_-a_2^{-2\nu} - \lambda_+\lambda_-c_0Ei(-iq\tau_2)\Big] \tag{4.122}$$

for $2\nu \neq 0$ and

$$c_0 = \mathcal{C}_p\mathcal{D}_{p'} + \mathcal{D}_p\mathcal{C}_{p'} \tag{4.123}$$

$$c_+ = \mathcal{C}_p\mathcal{C}_{p'} \tag{4.124}$$

$$c_- = \mathcal{D}_p\mathcal{D}_{p'} \tag{4.125}$$

and $\lambda_\pm = -\frac{1}{2} \pm (\nu = \sqrt{\frac{1}{4} - \frac{m^2}{H^2}})$ which can be either real or complex or zero, depending on its mass when compared to the expansion rate H.

4.5.3 Small Mass: $m < \frac{H}{2}$

When $m < \frac{H}{2}$, ν is real. Therefore,

$$\mathcal{T} = H^3 Re\int_{-\infty}^{t} dt_2(-2i)\sin q\tau_2\Big[\lambda_+^2c_+^*a_2^{2\nu} + \lambda_-^2c_-^*a_2^{-2\nu} + \lambda_+\lambda_-c_0^*\Big] \tag{4.126}$$

$$\times\Big[\frac{\lambda_+^2}{2\nu}c_+a_2^{2\nu} - \frac{\lambda_-^2}{2\nu}c_-a_2^{-2\nu} - \lambda_+\lambda_-c_0Ei(-iq\tau_2)\Big] \tag{4.127}$$

We can see that there is no contribution to the terms proportional to $|c_+|^2$, and $|c_-|^2$ because they are all real. With the factor i in the integrand, the contribution is purely imaginary. Hence there is no contribution after taking the real part. Therefore, the terms that give non-zero result are

$$\mathcal{T} = -\frac{H^2\lambda_+\lambda_-}{\nu} Re \int_{-\infty}^{\tau} \frac{d\tau_2}{\tau_2} i \sin q\tau_2 \tag{4.128}$$

$$\Big[2\nu\Big(\frac{\lambda_+^2 c_+^* c_0}{(-H\tau_2)^{2\nu}} + \frac{\lambda_-^2 c_-^* c_0}{(-H\tau_2)^{-2\nu}} - \lambda_+\lambda_-|c_0|^2\Big) Ei(-iq\tau_2) \tag{4.129}$$

$$-2\lambda_+\lambda_- i Im(c_-^* c_+) - \frac{\lambda_+^2 c_0^* c_+}{(-H\tau_2)^{2\nu}} + \frac{\lambda_-^2 c_0^* c_-}{(-H\tau_2)^{-2\nu}}\Big] \tag{4.130}$$

The integral above is still too complicated. However, when $0 \ll 2\nu < 1$, $\sin q\tau_2 \rightarrow q\tau_2 = -\frac{q}{a_2 H}$, and $Ei(-ix) \simeq \ln x + \gamma_E - \frac{i\pi}{2} \simeq \ln x$ when $x \rightarrow 0$ the result of time integral is less than t^2 or $(\ln a(t))^2$ at late time. Therefore, the dominant contribution comes from the first term of the integrand. Hence,

$$\mathcal{T} \rightarrow 2qH\lambda_-\lambda_+^3 Im(c_+^* c_0)\frac{(1-\eta\ln q\tau)}{a^\eta \eta^2} \tag{4.131}$$

for $0 \ll 2\nu < 1$. Therefore, the correlation function due to massive vector field loop is

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}\Big\langle \zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle \rightarrow -\frac{12(2\pi)^3\lambda_-\lambda_+^3(8\pi G H^2)^2 H}{q^5 a^\eta} \tag{4.132}$$

$$\times\frac{(1-\eta\ln q\tau)}{\eta^2}\int d^3p d^3p' \delta^3(\mathbf{q}+\mathbf{p}+\mathbf{p}') Im(c_+^* c_0) \tag{4.133}$$

To calculate what is $Im(c_+^* c_0)$, we see from eqs. (4.93) and (4.94) that $C_p^* D_p$ is p-independent. Therefore,

$$Im(c_+^* c_0) = \frac{(2H)^{2\nu}\Gamma^2(\nu)}{8\nu(2\pi)^7 H^2}\Big[\frac{1}{p^{2\nu}} + \frac{1}{p'^{2\nu}}\Big] \tag{4.134}$$

where we use $\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin \pi x}$ and

$$Im(\mathcal{C}_p^* \mathcal{D}_p) = \frac{1}{4(2\pi)^3 H\nu} \quad (4.135)$$

$$|\mathcal{C}_p|^2 = \frac{\pi\Gamma^2(\nu)}{(2\pi)^5 H}\Big(\frac{2H}{p}\Big)^{2\nu} \quad (4.136)$$

Hence,

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x'})}\Big\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle_{loop,m<\frac{H}{2}} = \quad (4.137)$$

$$-\frac{3(2)^{2\nu}\Gamma^2(\nu)\lambda_-\lambda_+^3(8\pi GH^2)^2}{2(2\pi)^4 H^{1-2\nu}q^5 a^{1-2\nu}}\frac{(1+(2\nu-1)\ln q\tau)}{\nu(1-2\nu)^2} \quad (4.138)$$

$$\int d^3p d^3p' \delta^3(\mathbf{q}+\mathbf{p}+\mathbf{p}')\Big[\frac{1}{p^{2\nu}}+\frac{1}{p'^{2\nu}}\Big] \quad (4.139)$$

To determine the momentum dependence q of the spectrum, we integrate over internal momentum p, p' circulated inside the loop. Note that, in general,

$$\int d^3p \int d^3p' \delta^3(\mathbf{p}+\mathbf{p'}+\mathbf{q}) f = \frac{2\pi}{q}\int_0^\infty pdp \int_{|p-q|}^{p+q} p'dp' f \quad (4.140)$$

and

$$\int_{|p-q|}^{p+q} p'dp' f(p,p',q) = F(p+q) - F(|p-q|) \quad (4.141)$$

$$= 2q\frac{\partial F}{\partial p} + \mathcal{O}(q^3) \simeq 2pqf(p, p'=p, q) \quad (4.142)$$

To ensure the approximation of the mode solutions at low momentum in eq. (4.91) is still valid, we cut off the upper limit of p integral at q. Therefore,

$$\int d^3p \int d^3p' \delta^3(\mathbf{p}+\mathbf{p'}+\mathbf{q}) f = 8\pi \int_0^q dp p^{2-2\nu} = \frac{8\pi q^{3-2\nu}}{3-2\nu} \quad (4.143)$$

Substituting equation above into eq. (4.137), we have the momentum spectrum as

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x'})}\Big\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle = \tag{4.144}$$

$$-\frac{24\pi(2)^{2\nu}\Gamma^2(\nu)\lambda_-\lambda_+^3(8\pi GH^2)^2}{4\pi(2\pi)^3H^{1-2\nu}q^{2+2\nu}a^{1-2\nu}}\frac{(1+(2\nu-1)\ln q\tau)}{\nu(3-2\nu)(1-2\nu)^2} \tag{4.145}$$

where $0 \ll 2\nu = \sqrt{1-\frac{4m^2}{H^2}} < 1$ (we will consider the case when $2\nu \to 0$ in the next section). By defining $\eta \equiv 1-2\nu$ as positive and small, we can rewrite the equation above in a more compact form as

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}\Big\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle = \tag{4.146}$$

$$-\frac{24\Gamma^2(\nu)\lambda_-\lambda_+^3(8\pi GH^2)^2}{(2\pi)^3(2Ha(t))^\eta}\frac{(1-\eta\ln q\tau)}{(1-\eta)(2+\eta)\eta^2q^{3-\eta}} \tag{4.147}$$

where

$$2\nu = \sqrt{1-\frac{4m^2}{H^2}} \simeq 1-\frac{2m^2}{H^2} \tag{4.148}$$

or

$$\eta \equiv 1-2\nu \simeq \frac{2m^2}{H^2} \tag{4.149}$$

We see that the departure from scale invariance is still small.

4.5.4 Critical Mass: $m\ \frac{H}{2}$

When $m = \frac{H}{2}$, we have $\nu = 0$ and $\lambda_+ = \lambda_- = -\frac{1}{2}$. Eq. (4.120) becomes

$$\begin{aligned}\int_{-\infty}^{t_2} dt_1 a_1 e^{-iq\tau_1}\dot{\mathcal{A}}_p(t_1)\dot{\mathcal{A}}_{p'}(t_1) &= \frac{H^2}{4}\Big(\mathcal{C}_p+\mathcal{D}_p\Big)\Big(\mathcal{C}_{p'}+\mathcal{D}_{p'}\Big)\int_{-\infty}^{t_2} dt_1 e^{-iq\tau_1} &(4.150)\\ &= -\frac{H}{4}\Big(\mathcal{C}_p+\mathcal{D}_p\Big)\Big(\mathcal{C}_{p'}+\mathcal{D}_{p'}\Big)Ei(-iq\tau_2) &(4.151)\end{aligned}$$

Substituting equation above into eq. (4.116), we therefore have

$$\mathcal{T} = -\frac{H^3}{16}|\mathcal{C}_p+\mathcal{D}_p|^2|\mathcal{C}_{p'}+\mathcal{D}_{p'}|^2 Re\int_{-\infty}^{t} dt_2(e^{-iq\tau_2}-e^{iq\tau_2})Ei(-iq\tau_2) \quad (4.152)$$

$$= -\frac{H^2}{8}|\mathcal{C}_p+\mathcal{D}_p|^2|\mathcal{C}_{p'}+\mathcal{D}_{p'}|^2\int_{-\infty}^{0}\frac{d\tau_2}{\tau_2}\sin q\tau_2 Si(q\tau_2) \quad (4.153)$$

$$= \frac{\pi^2 H^2}{64}|\mathcal{C}_p+\mathcal{D}_p|^2|\mathcal{C}_{p'}+\mathcal{D}_{p'}|^2 \quad (4.154)$$

We see from eqs. (4.93) and (4.94) that the coefficients are all momentum independent when $\nu=0$. Since

$$\Big(\mathcal{C}_q+\mathcal{D}_q\Big)|_{\nu=0,m=\frac{H}{2}} = \frac{0}{0} \quad (4.155)$$

we need to use l' Hospital's rule. From eqs. (4.93) and (4.94), we have

$$\lim_{\nu\to 0}\Big(\mathcal{C}_q+\mathcal{D}_q\Big) = -\frac{i\sqrt{\pi}e^{\frac{i\pi}{4}}}{2(2\pi)^{\frac{3}{2}}\sqrt{H}}\lim_{\nu\to 0}\Big[\frac{e^{\frac{i\pi\nu}{2}}(\frac{2H}{q})^{\nu}}{\sin\nu\pi\Gamma(1-\nu)} - \frac{e^{-\frac{i\pi\nu}{2}}(\frac{2H}{q})^{-\nu}}{\sin\nu\pi\Gamma(1+\nu)}\Big] \quad (4.156)$$

Note that

$$\lim_{\nu\to 0}\frac{e^{\frac{i\pi\nu}{2}}(\frac{2H}{q})^{\nu}}{\sin\nu\pi\Gamma(1-\nu)} = \lim_{\nu\to 0}\frac{e^{\frac{i\pi\nu}{2}}(\frac{i\pi}{2}(\frac{2H}{q})^{\nu}+\nu(\frac{2H}{q})^{\nu-1})}{\pi\cos\nu\pi\Gamma(1-\nu)-\sin\nu\pi\psi(1-\nu)\Gamma(1-\nu)} \quad (4.157)$$

$$= \frac{i}{2} \quad (4.158)$$

Similarly,

$$\lim_{\nu\to 0}\frac{e^{-\frac{i\pi\nu}{2}}(\frac{2H}{q})^{-\nu}}{\sin\nu\pi\Gamma(1+\nu)} = -\frac{i}{2} \quad (4.159)$$

Therefore,

$$\lim_{\nu\to 0}\Big(\mathcal{C}_q+\mathcal{D}_q\Big) = \frac{\sqrt{\pi}e^{\frac{i\pi}{4}}}{2(2\pi)^{\frac{3}{2}}\sqrt{H}} \quad (4.160)$$

We see that the coefficients are q-independent. Hence, eq. (4.152) becomes

$$\mathcal{T} = \frac{\pi^4}{1024(2\pi)^6} \tag{4.161}$$

Therefore the correlation function in eq.(4.114) becomes

$$\begin{aligned}
\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}\Big\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle &\rightarrow -\frac{3\pi^2(8\pi GH^2)^2}{1024q^6}\int_0^q p^2dp && (4.162)\\
&= -\frac{\pi^2(8\pi GH^2)^2}{1024q^3} && (4.163)
\end{aligned}$$

4.5.5 Large Mass: $m > \frac{H}{2}$

For $m > \frac{H}{2}$, $\lambda_\pm$ are complex conjugate of each other. We define $\lambda_+ \equiv \lambda = -\frac{1}{2} + ir$, $\lambda_- = \lambda^* = -\frac{1}{2} - ir$ where $\nu = ir$ and $r \equiv \sqrt{|\frac{m^2}{H^2} - \frac{1}{4}|}$. Therefore, eq. (4.120) becomes

$$\begin{aligned}
\int_{-\infty}^{t_2} dt_1 a_1 e^{-iq\tau_1}\dot{\mathcal{A}}_p(t_1)\dot{\mathcal{A}}_{p'}(t_1) &\rightarrow \frac{H}{2ir}\Big[\lambda^2 c_+ a_2^{2ir} - \lambda^{*2}c_- a_2^{-2ir} && (4.164)\\
&\quad -2ir|\lambda|^2 c_0 Ei(-iq\tau_2)\Big] && (4.165)
\end{aligned}$$

We see that the time components of the first two terms are complex conjugate of each other with different constant coefficients. Integrating over time t_2 gives

$$\begin{aligned}
\mathcal{T} &= -\frac{H^3}{r}Re\int_{-\infty}^{t} dt_2 \sin q\tau_2\Big[\lambda^{*2}c_+^* e^{-2irHt_2} + \lambda^2 c_-^* e^{2irHt_2} + |\lambda|^2 c_0^*\Big] && (4.166)\\
&\quad \times\Big[\lambda^2 c_+ e^{2irHt_2} - \lambda^{*2}c_- e^{-2irHt_2} - 2ir|\lambda|^2 c_0 Ei(-iq\tau_2)\Big] && (4.167)\\
&= \frac{|\lambda|^4 H^2}{r}\int_{-\infty}^{0}\frac{d\tau_2}{\tau_2}\sin q\tau_2\Big(|c_+|^2 - |c_-|^2 - 2r|c_0|^2 Siq\tau_2\Big) && (4.168)
\end{aligned}$$

where the other terms vanish in the limit of $t \to \infty$ because of the oscillating behavior of the integrand $e^{\pm iHt_2}$. We can integrate further with the use of Mathematica

$$\int_{-\infty}^{0} \frac{d\tau_2}{\tau_2} \sin q\tau_2 Siq\tau_2 = -\frac{\pi^2}{8} \tag{4.169}$$

$$\int_{-\infty}^{0} \frac{d\tau_2}{\tau_2} \sin q\tau_2 = \frac{\pi}{2} \tag{4.170}$$

Therefore, eq. (4.166) becomes

$$\mathcal{T} = \frac{\pi|\lambda|^4 H^2}{4r}\Big[2(|c_+|^2 - |c_-|^2) + \pi r|c_0|^2\Big] \tag{4.171}$$

We now need to calculate what $|c_{\pm,0}|^2$ are. From eq. (4.123), we have

$$|c_+|^2 = |\mathcal{C}_p|^2|\mathcal{C}_{p'}|^2 \tag{4.172}$$

$$|c_-|^2 = |\mathcal{D}_p|^2|\mathcal{D}_{p'}|^2 \tag{4.173}$$

$$|c_0|^2 = |\mathcal{C}_p\mathcal{D}_{p'} + \mathcal{D}_p\mathcal{C}_{p'}|^2 \tag{4.174}$$

From eqs. (4.93) and (4.94), we have

$$|\mathcal{C}_q|^2 = \frac{\pi}{4(2\pi)^3 H|\sin\pi\nu|^2}\frac{e^{-\pi r}}{|\Gamma(1-\nu)|^2} \tag{4.175}$$

and

$$|\mathcal{D}_q|^2 = \frac{\pi}{4(2\pi)^3 H|\sin\nu\pi|^2}\frac{e^{\pi r}}{|\Gamma(1+\nu)|^2} \tag{4.176}$$

We can see that $|\mathcal{C}_q|^2$ and $|\mathcal{D}_q|^2$ are *momentum independent* because $\nu \equiv ir$ is purely imaginary in the large mass limit when $m > \frac{H}{2}$. Therefore,

$$|c_+|^2 - |c_-|^2 = \frac{\pi^2(e^{-2\pi r} - e^{2\pi r})}{16(2\pi)^6 H^2|\sin\nu\pi|^4|\Gamma(1+\nu)|^4} \tag{4.177}$$

Using $|\Gamma(1+ir)|^2 = |\Gamma(1-ir)|^2 = \frac{\pi r}{\sinh \pi r}$ and $\sin ir = i \sinh r$ for real r, the equation above is simplified as

$$|c_+|^2 - |c_-|^2 = -\frac{\coth \pi r}{4(2\pi)^6 r^2 H^2} \tag{4.178}$$

where we use $\sinh 2x = 2 \sinh x \cosh x$. Also,

$$|c_0|^2 = \frac{\cos^2(r \ln \frac{p}{p'})}{4(2\pi)^6 r^2 H^2 \sinh^2 \pi r} \tag{4.179}$$

Notice that the coefficients $|c_+|^2 - |c_-|^2$ are completely momentum independent and $|c_0|^2$ is nearly momentum independent ($\ln \frac{|\mathbf{p}|}{|\mathbf{p}+\mathbf{q}|} \to 0$ when $q \to 0$). Substituting eqs. (4.178), and (4.179) into eq. (4.171), we have

$$\mathcal{T} \;\; = \;\; \frac{\pi |\lambda|^4}{8r(2\pi)^6} \Big[-\frac{\coth \pi r}{r^2} + \frac{\pi \cos^2(r \ln \frac{p}{p'})}{2r \sinh^2 \pi r} \Big] \tag{4.180}$$

From eqs. (4.114) and (4.180), we have

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x'})} \Big\langle \zeta(\mathbf{x},t)\zeta(\mathbf{x}',t) \Big\rangle_{loop} = -\frac{3\pi(8\pi G H^2)^2 |\lambda|^4}{32 r (2\pi)^3 q^6} \int d^3p d^3p' \tag{4.181}$$

$$\times \delta^3(\mathbf{q}+\mathbf{p}+\mathbf{p}') \Big[-\frac{\coth \pi r}{r^2} + \frac{\pi \cos^2(r \ln \frac{p}{p'})}{2r \sinh^2 \pi r} \Big] \tag{4.182}$$

Note that, in general,

$$\int d^3p \int d^3p' \delta^3(\mathbf{p}+\mathbf{p}'+\mathbf{q}) f = \frac{2\pi}{q} \int_0^\infty p dp \int_{|p-q|}^{p+q} p' dp' f \tag{4.183}$$

and

$$\int_{|p-q|}^{p+q} p' dp' f(p,p',q) = F(p+q) - F(|p-q|) \tag{4.184}$$

$$= 2q \frac{\partial F}{\partial p} + \mathcal{O}(q^3) \simeq 2pq f(p, p'=p, q) \tag{4.185}$$

We cut off the momentum integral over p at $\Lambda q \simeq q$ so that the approximation of the massive vector mode solutions at low momentum in eq. (4.91) is still valid. Therefore, the result of the momentum integrals p, p' gives the momentum dependence as q^3, which cancels with the q^{-6} factor in eq. (4.181). We therefore have the approximated scale invariant spectrum as

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x'})} \Big\langle \zeta(\mathbf{x},t)\zeta(\mathbf{x}',t) \Big\rangle_{loop, m>\frac{H}{2}} = \frac{3\pi^2(8\pi GH^2)^2|\lambda|^4}{8r^3(2\pi)^3q^3} \coth \pi r \qquad (4.186)$$

where $\lambda = -\frac{1}{2} + ir$ and $r = \sqrt{|\frac{m^2}{H^2} - \frac{1}{4}|}$.

Chapter 5

Conformal Scalar Field

Don't accept something:

because you have heard it many times;

because it has been believed traditionally;

because it is believed by a large number of people;

because it is in accordance with your scripture;

because it seems logical;

because it is in line with your own beliefs;

because it is proclaimed by your teacher, who has an attractive personality and for whom you have great respect.

Accept it only after you have realized it yourself at the experiential level and have found it to be wholesome and beneficial to one and all. Then, not only accept it but also live up to it. Gotama

We have learned that the spectrums of massless minimal coupled scalar, massless fermion, and massless vector fields loops *all* go as $(8\pi GH^2)^2 q^{-3} \ln q$. We like to investigate whether this is still true for conformal scalar loop.

The full action considered during inflation is

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_g + \mathcal{L}_\varphi + \mathcal{L}_c && (5.1)\\
&= -\frac{1}{2}\sqrt{-g}[\frac{1}{8\pi G}R + g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + 2V(\varphi) + g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - \xi R\chi^2] && (5.2)
\end{aligned}$$

where φ is an inflaton , χ is an additional conformal scalar matter in which $\langle\chi\rangle = 0$, and $\xi = \frac{1}{6}, 0$ for conformal and minimal couplings respectively. We consider this to see how the conformal scalar affects the spectrum $\langle\zeta\zeta\rangle$ through the interaction with gravity. From the full action in eq. (5.1), we can expand the gravitational, minimal coupled scalar (inflaton) and conformal scalar fields as

$$\begin{aligned}
g_{\mu\nu} &= \bar{g}_{\mu\nu} + \delta g_{\mu\nu} && (5.3)\\
\varphi &= \bar{\varphi} + \delta\varphi && (5.4)\\
\chi &= 0 + \delta\chi && (5.5)
\end{aligned}$$

5.1 Field Equation and Its Solution

To arrive at the field equation of the conformal scalar field, we need the action up to the second order of the field fluctuation which is

$$\mathcal{L}^{(2)}_\chi = \frac{a^3}{2}\Big(\dot{\chi}^2 - \frac{(\partial_i\chi)^2}{a^2} - 12\xi H^2\chi^2\Big) \tag{5.6}$$

where $\bar{R} = -12H^2$. Varying the second order of the action with respect to χ, we have the field equation of the conformal scalar field as

$$\ddot{\chi}_q + 3H\dot{\chi}_q + \Big(\frac{q^2}{a^2} + 12\xi H^2\Big)\chi_q = 0 \tag{5.7}$$

Notice that if there is no extra term $12\xi H^2\chi$ (or $\xi = 0$), this is just a minimal coupled massless scalar, in which the dominant solution approaches constant at late time. It is known that the massless minimal coupled scalar produces a scale free spectrum. We like to investigate the momentum dependence of the power spectrum due to conformal scalar loop here. Eq. (5.7) can be solved exactly by re-scaling field $\chi \equiv u/a$. Hence, for conformal coupling $\xi = \frac{1}{6}$

$$u_q'' + \Big(q^2 - \frac{a''}{a} + 2H^2a^2\Big)u_q = 0 \tag{5.8}$$

where u' denotes the time derivative with respect to conformal time $\tau \equiv \int_t^\infty \frac{dt'}{a(t')}$. During inflation, $a \simeq -\frac{1}{H\tau}$, therefore, the last two terms of eq. (5.8) are cancelled. We arrive at a simple field equation of conformal scalar field as

$$u_q'' + q^2 u_q = 0 \tag{5.9}$$

Therefore, the solution to the equation above is just a simple plane wave valid to all wavelengths

$$\chi_q(t) = u_q(t)/a(t) = \frac{1}{(2\pi)^{\frac{3}{2}}a(t)\sqrt{2q}}e^{-iq\tau} \tag{5.10}$$

where we choose the constant coefficients to match with the positive mode solution at inside horizon. Since the time order product of the two field operators is the usual in-out Feynman propagator (or the $R-R$ propagator in in-in theories), its plane wave solution in eq. (5.10) gives the free field propagator as that in flat space. This arises due to the conformal flatness of the theories. As seen in the previous chapters, the solutions of massless fermion and massless gauge fields are also plane wave, apart from the different power of a and spinor and polarization factors. From eq. (5.10), the conformal scalar field correlation function to leading order is

$$\Big\langle \chi(\mathbf{x},t)\chi(\mathbf{x}',t)\Big\rangle = \int d^3q e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}|\chi_q(t)|^2 \tag{5.11}$$

$$= \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')} \frac{1}{2qa^2(t)} \tag{5.12}$$

We see that its momentum dependence is far from scale invariance at the classical level. However, we never observed the product of the scalar fields in CMB anisotropy but rather the correlation function of the temperature or density fluctuation which is related to the conserved quantity ζ. Therefore, we study how the conformal scalar field affects the observable power spectrum $\langle\zeta\zeta\rangle$ via the gravitational interactions at the quantum level.

5.2 Interaction Vertices

As seen from the previous section, the second order of the field fluctuations gives a free field equation and hence the propagators via its solutions. The cubic and higher order terms give the interaction vertices after expanding all field fluctuations. For example, the trilinear vertices due to conformal scalar χ and gravity ζ are

$$\mathcal{L}_{\zeta\chi\chi} = -\frac{a}{2}\zeta(\partial_i\chi)^2 + a\partial_i\Big(\frac{\zeta}{H} - \epsilon H a^2 \nabla^{-2}\dot{\zeta}\Big)\dot{\chi}\partial_i\chi \tag{5.13}$$

$$-\frac{a}{2H}\dot{\zeta}(\partial_i\chi)^2 - \frac{a^3}{2H}\dot{\zeta}\dot{\chi}^2 + \frac{3a^3}{2}\zeta\dot{\chi}^2 \tag{5.14}$$

$$-a^3\Big(3H^2\zeta + H\dot{\zeta} - \frac{\delta R}{12}\Big)\chi^2 \tag{5.15}$$

where the last line above contains the additional terms from the massless minimal coupled scalar. Those terms in the last line arise from the conformal term which is $\frac{\sqrt{-g}}{12}R\chi^2$. We see that the interactions above are rather complicated. However many terms are cancelled via field equation and removed by the field re-definition of ζ[2]. Therefore, the interaction vertices can be written in a more compact form as derived in appendix as

$$H_{\zeta\chi\chi}(t) = -\int d^3x \epsilon H a^5 (T^{00} + a^2 T^{ii}) \nabla^{-2}\dot{\zeta} \tag{5.16}$$

where $T^{\mu\nu}$ in the equation above is the energy momentum tensor at the second order of arbitrary matter. By definition of energy momentum tensor,

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_\chi}{\delta g_{\mu\nu}} \tag{5.17}$$

The combination of time and space components of energy momentum tensor is [14]

$$\begin{aligned} T^{00} + a^2 T^{ii} &= 2(1-3\xi)\dot{\chi}^2 + 12\xi^2 H^2 \chi^2 + \frac{2\xi}{a^2}\chi_{;i}^2 - 2\xi\chi\Big(\chi_{;00} + \frac{\chi_{;ii}}{a^2}\Big) &(5.18) \\ &\quad -\xi\chi^2\Big(\bar{R}_{00} + \frac{\bar{R}_{ii}}{a^2} - \bar{R} + 3\xi\bar{R}\Big) &(5.19) \end{aligned}$$

We can check that $T^{00} + a^2 T^{ii} = 2\dot{\chi}^2$ for minimal coupling $\xi = 0$. For conformal coupling $\xi = \frac{1}{6}$, we have

$$T^{00} + a^2 T^{ii} = \dot{\chi}^2 + \frac{1}{3}\Big(\frac{(\partial_i\chi)^2}{a^2} - 2\chi\ddot{\chi} - H^2\chi^2\Big) \tag{5.20}$$

where the conformal field equation (5.7) is used and

$$\bar{R}_{00} + \frac{\bar{R}_{ii}}{a^2} - \bar{R} + 3\xi\bar{R} = 0 \tag{5.21}$$

due to $\bar{R}_{00} = 3H^2, \bar{R}_{ii} = -9a^2H^2$ and $\bar{R} = -12H^2$ in de-Sitter phase inflation. Therefore, the trilinear interaction Hamiltonian of conformal scalar χ and gravity ζ is

$$H_{\zeta\chi\chi}(t) = -\int d^3x \epsilon H a^5 \Big[\dot{\chi}^2 + \frac{1}{3}(\frac{(\partial_i\chi)^2}{a^2} - 2\chi\ddot{\chi} - H^2\chi^2)\Big]\nabla^{-2}\dot{\zeta} \tag{5.22}$$

5.3 Infrared Safe

By counting the power of a, we see that the interaction Hamiltonian in eqs. (5.13) or (5.22) has a term that contains the explicit factor of a^3. The reason is that ζ goes at most constant and $\dot{\zeta}$ goes as a^{-2} at late time. By looking at the conformal mode solution in eq. (5.10), both of χ and $\dot{\chi}$ go as a^{-1} at late time. Therefore, it appears that the interaction Hamiltonian in eq. (5.22) gives the power of a as $a^{5-2-2(1)=1}$. However, we need to consider how the conformal scalar field and its time derivative enter into the commutator(s) as well.

The commutator of two interaction-picture scalar fields at time t_1, t_2 is

$$\Big[\chi(\mathbf{x}_1, t_1), \chi(\mathbf{x}_2, t_2)\Big] = 2i \int d^3q e^{i\mathbf{q}\cdot(\mathbf{x}_1-\mathbf{x}_2)} Im\Big[\chi_q(t_1)\chi_q^*(t_2)\Big] \tag{5.23}$$

The commutator of a field and a field time derivative is

$$\Big[\chi(\mathbf{x}_1, t_1), \dot{\chi}(\mathbf{x}_2, t_2)\Big] = 2i \int d^3q e^{i\mathbf{q}\cdot(\mathbf{x}_1-\mathbf{x}_2)} Im\Big[\chi_q(t_1)\dot{\chi}_q^*(t_2)\Big] \tag{5.24}$$

The commutator of two field time derivative is

$$\Big[\dot{\chi}(\mathbf{x}_1, t_1), \dot{\chi}(\mathbf{x}_2, t_2)\Big] = 2i \int d^3q e^{i\mathbf{q}\cdot(\mathbf{x}_1-\mathbf{x}_2)} Im\Big[\dot{\chi}_q(t_1)\dot{\chi}_q^*(t_2)\Big] \tag{5.25}$$

From the solution of the conformal scalar field equation in eq. (5.10), we have

$$\dot{\chi}_q(t) = -\Big(H + \frac{iq}{a}\Big)\chi_q(t) \tag{5.26}$$

Hence,

$$\chi_q(t_1)\chi_q^*(t_2) = \frac{1}{2(2\pi)^3 q a_1 a_2} e^{-iq(\tau_1-\tau_2)} \tag{5.27}$$

$$\chi_q(t_1)\dot{\chi}^*_q(t_2) = -\frac{1}{2(2\pi)^3 q a_1 a_2}\Big(H - \frac{iq}{a_2}\Big)e^{-iq(\tau_1-\tau_2)} \tag{5.28}$$

$$\dot{\chi}_q(t_1)\dot{\chi}^*_q(t_2) = \frac{1}{2(2\pi)^3 q a_1 a_2}\Big(H + \frac{iq}{a_1}\Big)\Big(H - \frac{iq}{a_2}\Big)e^{-iq(\tau_1-\tau_2)} \tag{5.29}$$

Therefore,

$$Im\Big[\chi_q(t_1)\chi^*_q(t_2)\Big] = -\frac{1}{2(2\pi)^3 q a_1 a_2}\sin q(\tau_1-\tau_2) \tag{5.30}$$

$$Im\Big[\chi_q(t_1)\dot{\chi}^*_q(t_2)\Big] = \frac{1}{2(2\pi)^3 a_1 a_2^2}\cos q(\tau_1-\tau_2) \tag{5.31}$$

$$+\frac{H}{2(2\pi)^3 q a_1 a_2}\sin q(\tau_1-\tau_2) \tag{5.32}$$

$$Im\Big[\dot{\chi}_q(t_1)\dot{\chi}^*_q(t_2)\Big] = \frac{1}{2(2\pi)^3 q a_1 a_2}\Big[H(\frac{q}{a_1}-\frac{q}{a_2})\cos q(\tau_1-\tau_2) \tag{5.33}$$

$$-(H^2+\frac{q^2}{a_1 a_2})\sin q(\tau_1-\tau_2)\Big] \tag{5.34}$$

We see from equations above that the power of a in various commutators of eqs. (5.23), (5.24), and (5.25) all goes as $a^{-3}(t)$. Since the maximum number of explicit factor of a in any interaction is 3 and there are as many commutators as there are interactions, the total number of factors of a at late time is zero if the field outside commutators approaches constant (i.e ζ) or is less than zero if the field outside commutators goes as negative power of a (i.e most matter such as massive scalar, fermion, and gauge fields.). Therefore, the result of the N time integral in conformal scalar field theories will never be larger than $(\ln a)^N$.

5.4 The Momentum Dependence Loop Correlation Function

To calculate the loop spectrum in the commutator $[H_1, [H_2, Q]]$, we can use the general formula in eq.(2.46) with the replacement

$$\begin{aligned}
\psi^*\psi &\rightarrow \dot{\chi}^2 + \frac{1}{3}\Big(\frac{(\partial_i\chi)^2}{a^2} - 2\chi\ddot{\chi} - H^2\chi^2\Big) &(5.35)\\
\zeta_q(t_{1,2}) &\rightarrow -\dot{\zeta}_q(t_{1,2})/q^2 &(5.36)\\
V(t) &= -\epsilon H a^5(t) &(5.37)
\end{aligned}$$

Hence, the $\mathcal{Z}$ in eq. (2.48) is changed to be

$$\mathcal{Z} \rightarrow \frac{1}{q^4}\Big(\dot{\zeta}_q(t_1)\zeta_q^*(t)(\dot{\zeta}_q(t_2)\zeta_q^*(t) - \zeta_q(t)\dot{\zeta}_q^*(t_2))\Big) \tag{5.38}$$

and

$$\mathcal{M}_\chi \equiv \pi^*(t_2)\pi(t_1) \tag{5.39}$$

where

$$\pi(t) = \dot{\chi}_{p'}\dot{\chi}_p - \frac{1}{3}\chi_{p'}\ddot{\chi}_p - \frac{1}{3}\ddot{\chi}_{p'}\chi_p - \frac{1}{3}\Big(\frac{pp'}{a^2} + H^2\Big)\chi_{p'}\chi_p \tag{5.40}$$

With

$$\begin{aligned}
\dot{\chi}_q(t) &= -\Big(H + \frac{iq}{a}\Big)\chi_q &(5.41)\\
\ddot{\chi}_q(t) &= \Big[H^2 + 3iH\frac{q}{a} - \frac{q^2}{a^2}\Big]\chi_q &(5.42)
\end{aligned}$$

many terms are cancelled. Therefore,

$$\pi(t) = \frac{1}{3a^2}\Big[p^2 + p'^2 - 4pp'\Big]\chi_p(t)\chi_{p'}(t) \tag{5.43}$$

Therefore,

$$\begin{aligned}
\mathcal{M}_\chi &= \frac{1}{9a_1^2a_2^2}\Big(p^2 + p'^2 - 4pp'\Big)^2\chi_p^*(t_2)\chi_{p'}^*(t_2)\chi_p(t_1)\chi_{p'}(t_1) && (5.44)\\
&= \frac{1}{36(2\pi)^6a_1^4a_2^4pp'}\Big(p^2 + p'^2 - 4pp'\Big)^2 e^{-i(p+p')(\tau_1-\tau_2)} && (5.45)
\end{aligned}$$

From eq. (2.46), we have

$$\begin{aligned}
\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}\Big\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle_{loop} &= -8(2\pi)^9\int d^3pd^3p'\delta^3(\mathbf{q}+\mathbf{p}+\mathbf{p}') && (5.46)\\
&\int_{-\infty}^{t} dt_2\epsilon_2H_2a_2^5\int_{-\infty}^{t_2} dt_1\epsilon_1H_1a_1^5Re\Big(\mathcal{Z}\pi_2^*\pi_1\Big) && (5.47)
\end{aligned}$$

To calculate $\mathcal{Z}$, we use the solution of free field Mukhanov equation in interaction picture as

$$\zeta_q(t) = \zeta_q^o e^{-iq\tau}(1+iq\tau) \tag{5.48}$$

where

$$|\zeta_q^o|^2 = \frac{8\pi GH^2(t_q)}{2(2\pi)^3\epsilon(t_q)q^3} \tag{5.49}$$

Hence,

$$\mathcal{Z} = \frac{|\zeta_q^o|^4}{H_1H_2a_1^2a_2^2}\Big(e^{2iq\tau - iq(\tau_1+\tau_2)} - e^{iq(\tau_2-\tau_1)}\Big) \tag{5.50}$$

Substituting the $\mathcal{Z}$ part eq. (5.50) and matter part eq. (5.43) into eq. (5.46), we have the correlation function due to conformal scalar field loop as

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')} \Big\langle \zeta(\mathbf{x},t)\zeta(\mathbf{x}',t) \Big\rangle_{loop} = -\frac{2(2\pi)^3(8\pi GH^2)^2}{9q^6} \tag{5.51}$$

$$\times \int d^3p d^3p' \delta^3(\mathbf{q}+\mathbf{p}+\mathbf{p}') \Big[p^2 + p'^2 - 4pp'\Big]^2 \tag{5.52}$$

$$\times Re \int_{-\infty}^{\infty} dt_2 a_2 (e^{-iq\tau_2} - e^{iq\tau_2}) \chi_p^*(t_2) \chi_{p'}^*(t_2) \tag{5.53}$$

$$\times \int_{-\infty}^{t_2} dt_1 a_1 e^{-iq\tau_1} \chi_p(t_1) \chi_p(t_1) \tag{5.54}$$

With the mode solution in eq. (5.10), we have

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')} \Big\langle \zeta(\mathbf{x},t)\zeta(\mathbf{x}',t) \Big\rangle_{loop} = -\frac{(8\pi GH^2)^2}{18(2\pi)^3 q^6} \tag{5.55}$$

$$\times \int d^3p d^3p' \delta^3(\mathbf{q}+\mathbf{p}+\mathbf{p}') \frac{1}{pp'} \Big[p^2 + p'^2 - 4pp'\Big]^2 \mathcal{T} \tag{5.56}$$

where

$$\mathcal{T} = Re \int_{-\infty}^{0} d\tau_2 (e^{-iq\tau_2} - e^{iq\tau_2}) e^{i(p+p')\tau_2} \int_{-\infty}^{\tau_2} d\tau_1 e^{-i(p+p'+q)\tau_1} \tag{5.57}$$

$$= -\frac{1}{2q(p+p'+q)} \tag{5.58}$$

With the result of time integrations above, we have the loop correlation function as

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x'})} \Big\langle \zeta(\mathbf{x},t)\zeta(\mathbf{x}',t) \Big\rangle_{loop} = \frac{(8\pi GH^2)^2}{36(2\pi)^3 q^7} \tag{5.59}$$

$$\int d^3p d^3p' \delta^3(\mathbf{q}+\mathbf{p}+\mathbf{p}') \frac{\Big(p^2 + p'^2 - 4pp'\Big)^2}{pp'(p+p'+q)} \tag{5.60}$$

By power counting, we see that the momentum integral has *quartic* UV divergence when $p \to \infty$. As seen above, there is no infrared divergence when $p, p' \to 0$ for non-zero external momentum q. We use dimensional regularization to regulate UV divergence by following the calculation in [2] for massless minimal coupled scalar.

Note that eq. (5.59) can be written as

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}\Big\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle_{loop} = \frac{(8\pi GH^2)^2}{36(2\pi)^3q^7}\Big[\frac{2\pi}{q}\mathcal{K}(q)\Big] \tag{5.61}$$

where

$$\mathcal{K}(q) = \int_0^\infty dp \int_{|p-q|}^{|p+q|} dp' \frac{\Big(p^2+p'^2-4pp'\Big)^2}{(p+p'+q)} \tag{5.62}$$

The UV divergence of the integral above for $\delta = 0$ gives the pole term as

$$\frac{2\pi}{q}\mathcal{K}(q) \Rightarrow q^{4+\delta}F(\delta) \tag{5.63}$$

$$F(\delta) \to \frac{F_0}{\delta} + F_1 \tag{5.64}$$

Therefore, in the limit $\delta = 0$,

$$\frac{2\pi}{q}\mathcal{K}(q) = q^4\Big[F_0 \ln q + L\Big] \tag{5.65}$$

where L is a divergent constant. To calculate the coefficient F_0, we differentiate $\mathcal{K}(q)$ with respect to q six times after integrating over p'. This will reduce the power of p in the integrand. Then the result of integrating over p when $p \to \infty$ will be finite and give a coefficient number F_0. Using Mathematica for six derivatives, we have

$$\frac{d^6\mathcal{K}(q)}{dq^6} = -\frac{24}{q} \tag{5.66}$$

Therefore, we obtain the coefficient of $\ln q$ as

$$\mathcal{K}(q) = -\frac{q^5}{5}\ln q + L \tag{5.67}$$

where L is a divergent constant. Comparing this with eq. (5.65), we have

$$F_0 = -\frac{2\pi}{5} \tag{5.68}$$

Hence, we have the conformal scalar loop correlation function as

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')} \Big\langle \zeta(\mathbf{x},t)\zeta(\mathbf{x}',t) \Big\rangle_{loop} = -\frac{\pi(8\pi G H^2)^2}{90(2\pi)^3 q^3}\Big[\ln q + L\Big] \tag{5.69}$$

We see that it is nearly scale invariance and smaller than the classical result by factor $\frac{|\epsilon| H^2 \ln q}{M_{Pl}^2}$.

Chapter 6

The Quantum Nature of M_{Pl} Theories

What's to lament there in that?
For one who sees, as it actually is,
the pure arising of phenomena,
the pure seriality of fabrications,
there is no fear. Adhimutta

The important question for us is whether there is an additional suppression of factor G at the quantum level. If there is a vertex as large as M_{Pl}, it raises the possibility that the loop power spectrum $\langle\zeta\zeta\rangle$ could give a contribution in the order of the observed value. As described earlier, such theories can only exist in massive theories with $m \simeq M_{Pl}$ because inflaton φ and gravity $g_{\mu\nu}$ fluctuate around non-zero background and also contribute a factor of M_{Pl} to the second order of action. We investigate with the massive minimal coupled scalar field to see what happen when a coupling is as large as M_{Pl}.

As usual, the action we consider is the Einstein gravity, inflaton φ with an arbitrary potential $V(\varphi)$, and an unbroken symmetry scalar field $\langle\sigma\rangle = 0$ with a mass possibly as large as M_{Pl} such that

$$\mathcal{L}_{total} = -\frac{\sqrt{-g}}{16\pi G}R - \frac{1}{2}\sqrt{-g}\Big(g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + 2V(\varphi)\Big) + \mathcal{L}_\sigma \tag{6.1}$$

where

$$\mathcal{L}_\sigma = -\frac{1}{2}\sqrt{-g}\Big(g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma + m^2\sigma^2\Big) \tag{6.2}$$

Since m could be as large as M_{Pl}, it may affects the power spectrum through the interaction with gravitational fluctuation $\delta g_{\mu\nu}$. We investigate how large the loop spectrum can be when the mass and coupling are varied from nearly zero to M_{Pl}

6.1 Field Equation and Its Solution

For $\sigma_q \equiv a^{-1}u_q$, the scalar field equation in inflating universe is

$$u_q'' + \Big(q^2 + \frac{1}{\tau^2}(\frac{m^2}{H^2} - 2)\Big)u_q = 0 \tag{6.3}$$

We see that if $m^2 = 2H^2$, the field equation above is the same as the conformal field equation shown in the previous chapter. For a general mass m and a general momentum q, the field equation above is a type of Bessel's equation. The Bessel's equation is[11]

$$u_q'' + \Big(q^2 - \frac{4\nu^2 - 1}{4\tau^2}\Big)u_q = 0 \tag{6.4}$$

Therefore, the general solutions are

$$\sigma_q(t) = u_q(t)/a(t) = a^{-1}\Big(\mathcal{E}_q\sqrt{-\tau}H_\nu^{(1)}(-q\tau) + \mathcal{F}_q\sqrt{-\tau}H_\nu^{(2)}(-q\tau)\Big) \tag{6.5}$$

with

$$\nu^2 = \frac{9}{4} - \frac{m^2}{H^2} \tag{6.6}$$

Since we want the solution to match with the positive solution at deep inside horizon $e^{-i\omega\tau}$, only $H_\nu^{(1)}(x)$ but not $H_\nu^{(2)}(x)$ gives a factor of $e^{-i\omega\tau}$ in the large $|x|$ limit. Hence, the constant $\mathcal{F}_q = 0$ and

$$\sigma_q(t) = \frac{\mathcal{E}_q\sqrt{-\tau}}{a(t)} H_\nu^{(1)}(-q\tau) \tag{6.7}$$

The normalized constant $\mathcal{E}_q$ is chosen to match with the solution at deep inside horizon. At inside horizon, the positive frequency solutions are the same as that in flat space, which are,

$$\sigma_q(t) \rightarrow \frac{1}{(2\pi)^{\frac{3}{2}} a(t)\sqrt{2\omega_q}} \exp\Big(-i\int_{-\infty}^{\tau} \omega_q(\tau')d\tau'\Big) \tag{6.8}$$

where $\omega_q(\tau) \equiv \sqrt{q^2 + (ma)^2}$. With the property of Hankel's function in asymptotic limit,$|x| \rightarrow \infty$

$$H_\nu^{(1)}(x) \rightarrow \sqrt{\frac{2}{\pi x}} \exp\Big(i(x - \frac{\pi\nu}{2} - \frac{\pi}{4})\Big) \tag{6.9}$$

Since we now allow the existence of a mass term which could be either large or small, the normalized constants $\mathcal{E}_q$ could be a function of mass and may affect the result of the correlation function.

To determine the time independent coefficient $\mathcal{C}_q$, we match the solution with that of inside horizon. From eqs. (6.7), (6.8), and (6.9), we have the mass dependent coefficient $\mathcal{E}_q$ as

$$\mathcal{E}_q(m) = \frac{\sqrt{\pi}}{2(2\pi)^{\frac{3}{2}}} e^{\frac{i\pi}{4}(1+2\nu)} \tag{6.10}$$

Substituting eq. (6.10) into eq. (6.7), we obtain the solution of a massive scalar field in inflating universe with an arbitrary wavelength as

$$\sigma_q(t) \quad = \quad \frac{\sqrt{\pi}}{2(2\pi a)^{\frac{3}{2}}\sqrt{H}} e^{\frac{i\pi}{4}(1+2\nu)} H_\nu^{(1)}(-q\tau) \tag{6.11}$$

6.2 Interaction Vertices

Owing to the existence of mass, the maximum number of explicit factor a that can arise in the interaction action is a^3. For example, the trilinear interaction that we will use to calculate the loop spectrum is

$$H_{\zeta\sigma\sigma}(t) = -\int d^3x \epsilon H a^5 (2\dot{\sigma}^2 - m^2\sigma^2)\nabla^{-2}\dot{\zeta} + \dot{Y} \tag{6.12}$$

As stated earlier, the equation above came from the calculation of the energy momentum tensor of massive scalar field $T^{00} + a^2 T^{ii}$.

6.3 Late Time Behavior

The late time behavior of mode solution can be obtained by using the property of Hankel function for small $|x|$

$$H_\nu^{(1)} \to -\frac{i}{\sin\nu\pi}\Big(\frac{x^{-\nu}}{2^{-\nu}\Gamma(1-\nu)} - \frac{e^{-\nu\pi i}x^{\nu}}{2^{\nu}\Gamma(\nu+1)}\Big)\Big(1+\mathcal{O}(x)^2\Big) \tag{6.13}$$

From eqs. (6.11) and (6.13), the solution at outside horizon is

$$\sigma_q(t) \to C_q a^{\lambda_+} + D_q a^{\lambda_-} \tag{6.14}$$

where

$$\lambda_\pm = -\frac{3}{2} \pm \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \tag{6.15}$$

and

$$C_q = -\frac{i\sqrt{\pi}e^{\frac{i\pi}{4}+\frac{i\pi\nu}{2}}}{2(2\pi)^{\frac{3}{2}}\sqrt{H}\sin\nu\pi\Gamma(1-\nu)}\Big(\frac{2H}{q}\Big)^{\nu} \tag{6.16}$$

$$D_q = \frac{i\sqrt{\pi}e^{\frac{i\pi}{4}-\frac{i\pi\nu}{2}}}{2(2\pi)^{\frac{3}{2}}\sqrt{H}\sin\nu\pi\Gamma(1+\nu)}\Big(\frac{2H}{q}\Big)^{-\nu} \tag{6.17}$$

We see from equation above that σ_q does not really approach constant after horizon exit due to the existence of mass. By counting the power of a in eq. (6.14) only, σ_q approaches $a^{-\frac{3}{2}\pm\nu}$ where ν is either positive real or positive imaginary or zero depending on its mass when compared to the expansion rate. As analyzed in [3], the result can never be proportional to the positive power of a. However, if the vertex or its mass is as large as M_{Pl}, the loop spectrum may not be small as we previously thought. For example, its time derivative contributes the same power of a at late time with an additional factor $H\lambda_\pm$ such that

$$\dot{\sigma}_q(t) \to -H\Big(C_q\lambda_+a^{\lambda_+} + D_q\lambda_-a^{\lambda_-}\Big) \tag{6.18}$$

Since $\lambda_\pm$ contains the mass term and the mass term could be as large as M_{Pl}, it may affect the order of magnitude in the loop power spectrum.

6.4 Unequal Time Commutators of Fields with Planck's mass

For a very heavy mass scalar field $m \simeq M_{Pl} > \frac{3H}{2}$, the two independent solutions at late time are a complex conjugate of each other with different coefficients C_q and D_q such that

$$\sigma_q(t) \to C_qa^{\lambda} + D_qa^{\lambda^*} \tag{6.19}$$

where

$$\lambda = -\frac{3}{2} + \sqrt{\frac{9}{4} - \frac{M_{Pl}^2}{H^2}} \to -\frac{3}{2} + \frac{iM_{Pl}}{H} \tag{6.20}$$

For a massive scalar field, its time derivative contributes the same power of a at late time with additional factor $H\lambda$ such that

$$\dot{\sigma}_q(t) \to H\Big(C_q \lambda a^{\lambda} + D_q \lambda^* a^{\lambda^*}\Big) \tag{6.21}$$

Hence,

$$\Big[\sigma(\mathbf{x}_1, t_1), \sigma(\mathbf{x}_2, t_2)\Big] = 2i \int d^3q e^{i\mathbf{q}\cdot(\mathbf{x}_1 - \mathbf{x}_2)} Im\Big(|C_q|^2 a_1^{\lambda} a_2^{\lambda^*} + |D_q|^2 a_1^{\lambda^*} a_2^{\lambda}\Big) \tag{6.22}$$

$$\Big[\sigma(\mathbf{x}_1, t_1), \dot{\sigma}(\mathbf{x}_2, t_2)\Big] = 2iH \int d^3q e^{i\mathbf{q}\cdot(\mathbf{x}_1 - \mathbf{x}_2)} Im\Big(|C_q|^2 \lambda^* a_1^{\lambda} a_2^{\lambda^*} + |D_q|^2 \lambda a_1^{\lambda^*} a_2^{\lambda}\Big) \tag{6.23}$$

and

$$\Big[\dot{\sigma}(\mathbf{x}_1, t_1), \dot{\sigma}(\mathbf{x}_2, t_2)\Big] = 2iH^2 |\lambda|^2 \int d^3q e^{i\mathbf{q}\cdot(\mathbf{x}_1 - \mathbf{x}_2)} Im\Big(|C_q|^2 a_1^{\lambda} a_2^{\lambda^*} + |D_q|^2 a_1^{\lambda^*} a_2^{\lambda}\Big) \tag{6.24}$$

where the combination of the two cross terms are real and hence its imaginary part is zero such that

$$Im\Big(C_q D_q^* a_1^{\lambda} a_2^{\lambda} + C_q^* D_q a_1^{\lambda^*} a_2^{\lambda^*}\Big) = 0 \tag{6.25}$$

Since $|C_q|^2$ and $|D_q|^2$ are both real, eqs. (6.22), (6.23), and (6.24) become

$$\Big[\sigma(\mathbf{x}_1, t_1), \sigma(\mathbf{x}_2, t_2)\Big] = \frac{2i}{a_1^{\frac{3}{2}} a_2^{\frac{3}{2}}} \int d^3q e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})} \Big(|C_q|^2 - |D_q|^2\Big) Im\Big(a_1^{\frac{iM_{Pl}}{H}} a_2^{-\frac{iM_{Pl}}{H}}\Big) \tag{6.26}$$

$$\left[\sigma(\mathbf{x}_1,t_1),\dot{\sigma}(\mathbf{x}_2,t_2)\right] = -\frac{2iH}{a_1^{\frac{3}{2}}a_2^{\frac{3}{2}}}\int d^3q e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}\left(|C_q|^2-|D_q|^2\right)Im\left(\lambda^* a_1^{\frac{iM_{Pl}}{H}} a_2^{-\frac{iM_{Pl}}{H}}\right) \quad (6.27)$$

and

$$\left[\dot{\sigma}(\mathbf{x}_1,t_1),\dot{\sigma}(\mathbf{x}_2,t_2)\right] = H^2|\lambda|^2\left[\sigma(\mathbf{x}_1,t_1),\sigma(\mathbf{x}_2,t_2)\right] \quad (6.28)$$

$$= \left(\frac{9H^2}{4}+M_{Pl}^2\right)\left[\sigma(\mathbf{x}_1,t_1),\sigma(\mathbf{x}_2,t_2)\right] \quad (6.29)$$

The additional factor in the time derivative of a field, $-H\lambda = -H(-\frac{3}{2}+\frac{iM_{Pl}}{H})$, may affect the large power spectrum due to the M_{Pl} factor. However, we need to consider the mass dependent coefficients C_q and D_q to investigate if there is any other suppression.

As seen from eq. (6.16), the $|C_q|^2$ and $|D_q|^2$ is *momentum independent* because ν is purely imaginary in the large mass limit when $m \simeq M_{Pl} > \frac{3H}{2}$. Note that $|\Gamma(1+\nu)|^2 = |\Gamma(1-\nu)|^2 = \frac{\pi M_{Pl}}{H\sinh\frac{\pi M_{Pl}}{H}}$ for $\nu = \frac{iM_{Pl}}{H}$. Therefore,

$$|C_q|^2 \mp |D_q|^2 = \frac{\pi}{4(2\pi)^3 H|\sin\nu\pi|^2|\Gamma(1+\nu)|^2}\left(e^{-\frac{\pi M_{Pl}}{H}} \mp e^{\frac{\pi M_{Pl}}{H}}\right) \quad (6.30)$$

Using $\sin ix = i\sinh x$ for real x, all the $\sinh x$ terms are cancelled for the combination $|C_q|^2-|D_q|^2$. Hence, the equation above is simplified as

$$|C_q|^2 - |D_q|^2 = -\frac{1}{2(2\pi)^3 M_{Pl}} \quad (6.31)$$

$$|C_q|^2 + |D_q|^2 = \frac{1}{2(2\pi)^3 M_{Pl}}\tanh\frac{\pi M_{Pl}}{H} \quad (6.32)$$

We see that the coefficient factor in an unequal time commutator get suppressed by only a factor of M_{Pl}^{-1}. Substituting eq. (6.31) in eqs. (6.26), (6.27), and (6.28), we have the various commutators of M_{Pl} massive scalar field as

$$\Big[\sigma(\mathbf{x}_1,t_1),\sigma(\mathbf{x}_2,t_2)\Big] = -\frac{i}{a_1^{\frac{3}{2}}a_2^{\frac{3}{2}}(2\pi)^3 M_{Pl}}\int d^3q e^{i\mathbf{q}\cdot(\mathbf{x}_1-\mathbf{x}_2)} Im\Big(a_1^{\frac{iM_{Pl}}{H}}a_2^{-\frac{iM_{Pl}}{H}}\Big) \quad (6.33)$$

$$= -\frac{i\sin M_{Pl}(t_1-t_2)}{a_1^{\frac{3}{2}}a_2^{\frac{3}{2}}(2\pi)^3 M_{Pl}}\delta^{(3)}(\mathbf{x}_1-\mathbf{x}_2) \quad (6.34)$$

$$\Big[\sigma(\mathbf{x}_1,t_1),\dot\sigma(\mathbf{x}_2,t_2)\Big] = \frac{iH}{a_1^{\frac{3}{2}}a_2^{\frac{3}{2}}(2\pi)^3 M_{Pl}}\int d^3q e^{i\mathbf{q}\cdot(\mathbf{x}_1-\mathbf{x}_2)} Im\Big((-\frac{3}{2}-\frac{iM_{Pl}}{H})a_1^{\frac{iM_{Pl}}{H}}a_2^{-\frac{iM_{Pl}}{H}}\Big) \quad (6.35)$$

$$= -\frac{i\delta^{(3)}(\mathbf{x}_1-\mathbf{x}_2)}{a_1^{\frac{3}{2}}a_2^{\frac{3}{2}}(2\pi)^3}\Big(\frac{3H}{2M_{Pl}}\sin M_{Pl}(t_1-t_2)+\cos M_{Pl}(t_1-t_2)\Big) \quad (6.36)$$

and

$$\Big[\dot\sigma(\mathbf{x}_1,t_1),\dot\sigma(\mathbf{x}_2,t_2)\Big] = -\Big(\frac{9H^2}{4M_{Pl}}+M_{Pl}\Big)\frac{i\sin M_{Pl}(t_1-t_2)}{a_1^{\frac{3}{2}}a_2^{\frac{3}{2}}(2\pi)^3}\delta^{(3)}(\mathbf{x}_1-\mathbf{x}_2) \quad (6.37)$$

We can check that, in the equal time limit, $[\sigma,\sigma]=[\dot\varphi,\dot\varphi]=0$ and $[\sigma,\dot\sigma]\propto\delta^{(3)}(\mathbf{x}_1-\mathbf{x}_2)$ as expected. The fact that the coefficients $|C_q|^2$ and $|D_q|^2$ are q-independent at $m>\frac{3H}{2}$ makes the integral over momentum trivial. The commutator above shows the possibility that the loop spectrum can be large because of a coefficient constant in the order of M_{Pl}. However, there is also contribution of fields which are *not* in the commutator that may affect the final spectrum. Therefore, we explicitly calculate the loop spectrum as well as its momentum dependence in the next section.

6.5 One Loop Two-Point Function

When we evaluate the RHS of eq. (2.2), there are fields that are in the commutators as well as those that are not in the commutators because there can only be a pair of fields in each unequal time commutator and the interaction Hamiltonian contains more than two fields. We therefore need to consider how the mass affects the spectrum through the field *both* inside and outside commutators. We therefore consider an example of one-loop two point function below.

To calculate the momentum dependence of loop spectrum, we use eq. (2.46). We replace

$$
\begin{aligned}
\psi^*\psi &\rightarrow 2\dot{\sigma}^2 - m^2\sigma^2 && (6.38)\\
\zeta_q(t_{1,2}) &\rightarrow -\dot{\zeta}_q(t_{1,2})/q^2 && (6.39)\\
V(t) &= -\epsilon H a^5(t) && (6.40)
\end{aligned}
$$

where $\mathcal{Z}$ is still the same as that in eq. (5.38). For matter part σ, there are terms containing the four product of the fields, the four product of the fields time derivative, and the two cross terms. Therefore, eq. (2.51) becomes

$$
\begin{aligned}
\mathcal{M}_\sigma &= 4\dot{\sigma}_p(t_1)\dot{\sigma}_{p'}(t_1)\dot{\sigma}^*_p(t_2)\dot{\sigma}^*_{p'}(t_2) && (6.41)\\
&+m^4\sigma_p(t_1)\sigma_{p'}(t_1)\sigma^*_p(t_2)\sigma^*_{p'}(t_2) && (6.42)\\
&-2m^2\sigma_p(t_1)\sigma_{p'}(t_1)\dot{\sigma}^*_p(t_2)\dot{\sigma}^*_{p'}(t_2) && (6.43)\\
&-2m^2\dot{\sigma}_p(t_1)\dot{\sigma}_{p'}(t_1)\sigma^*_p(t_2)\sigma^*_{p'}(t_2) && (6.44)
\end{aligned}
$$

Note that if the minimal coupled scalar is massless, only the first line contributes and the momentum dependence loop spectrum goes as $q^{-3}\ln q$ [2]. We like to investigate whether this is still true even when the scalar field has an arbitrary mass. Especially, when $m \simeq M_{Pl}$, there is the possibility seen in eq. (6.41) that the loop power spectrum can be large.

Since we will integrate over time t_1 before time t_2, eq. (6.41) can be grouped as

$$
\mathcal{M}_\sigma = \pi^*(t_2)\pi(t_1) \qquad (6.45)
$$

where

$$\pi(t) \equiv 2\dot{\sigma}_p(t)\dot{\sigma}_{p'}(t) - m^2\sigma_p(t)\sigma_{p'}(t) \tag{6.46}$$

$$= (2\lambda_+^2 H^2 - m^2)C_p C_{p'} a^{2\lambda_+} + (2\lambda_-^2 H^2 - m^2)D_p D_{p'} a^{2\lambda_-} \tag{6.47}$$

$$+(2\lambda_+\lambda_- H^2 - m^2)(C_p D_{p'} + D_p C_{p'})a^{\lambda_+ + \lambda_-} \tag{6.48}$$

Using $\lambda_\pm = -\frac{3}{2} \pm \nu$ in which $\nu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}$, we have the integrand of the matter part at an arbitrary mass as

$$\pi(t) = \frac{1}{a^3}\Big((2\lambda_+^2 H^2 - m^2)C_p C_{p'} a^{2\nu} + (2\lambda_-^2 H^2 - m^2)D_p D_{p'} a^{-2\nu} \tag{6.49}$$

$$+(2\lambda_+\lambda_- H^2 - m^2)(C_p D_{p'} + D_p C_{p'})\Big) \tag{6.50}$$

From eqs. (2.46), (5.49) (5.50), we have the correlation function as

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x'})}\Big\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle_{loop} = -\frac{2(2\pi)^3(8\pi G H^2(t_q))^2}{q^6} \tag{6.51}$$

$$\times\int d^3p d^3p' \delta^3(\mathbf{q}+\mathbf{p}+\mathbf{p}')\mathcal{T} \tag{6.52}$$

where

$$\mathcal{T} = Re\int_{-\infty}^{\infty} dt_2 a_2^3(e^{-iq\tau_2} - e^{iq\tau_2})\pi_2^* \int_{-\infty}^{t_2} dt_1 a_1^3 e^{-iq\tau_1}\pi_1 \tag{6.53}$$

The result in eqs. (6.49) and (6.51) is valid for an arbitrary mass m. We will consider what happen to the spectrum when the mass is as large as Planck's mass. However let us first consider the critical mass.

6.5.1 Critical Mass: $m = \frac{3H}{2}$

For $m = \frac{3H}{2}$, $\nu = 0$ and $\lambda_{\pm} = -\frac{3}{2}$. Eq. (6.49) has the simple form of

$$\pi(t)|_{m=\frac{3H}{2}} = \frac{9H^2}{4a^3}(C_p + D_p)(C_{p'} + D_{p'}) \tag{6.54}$$

Therefore,

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x'})}\Big\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x'},t)\Big\rangle_{loop} = -\frac{81(2\pi)^3H^4(8\pi GH^2)^2}{8q^6}\int d^3pd^3p' \tag{6.55}$$
$$\delta^3(\mathbf{q}+\mathbf{p}+\mathbf{p'})|C_p + D_p|^2|C_{p'} + D_{p'}|^2\mathcal{T} \tag{6.56}$$

where

$$\mathcal{T} = Re\int_{-\infty}^{t} dt_2(e^{-iq\tau_2} - e^{iq\tau_2})\int_{-\infty}^{t_2} dt_1 e^{-iq\tau_1} \tag{6.57}$$

Since

$$\int_{-\infty}^{t_2} dt_1 e^{-iq\tau_1} = -\frac{Ei(-iq\tau_2)}{H} \tag{6.58}$$

Therefore,

$$\mathcal{T} = -\frac{2}{H^2}\int_{-\infty}^{0}\frac{d\tau_2}{\tau_2}\sin q\tau_2 Si(q\tau_2) \tag{6.59}$$
$$= \frac{\pi^2}{4H^2} \tag{6.60}$$

Similar to massive vector field case, when $\nu = 0$, eq. (6.16) gives

$$C_q + D_q = \frac{0}{0} \tag{6.61}$$

Therefore, we need to use l' Hospital's rule in a similar way as in the critical massive vector field. From eq. (6.16), we have

$$\lim_{\nu\to 0}\frac{e^{\frac{i\pi\nu}{2}}(\frac{2H}{q})^{\nu}}{\sin\nu\pi\Gamma(1-\nu)} = -\lim_{\nu\to 0}\frac{e^{-\frac{i\pi\nu}{2}}(\frac{2H}{q})^{-\nu}}{\sin\nu\pi\Gamma(1+\nu)} \tag{6.62}$$

$$= \frac{i}{2} \tag{6.63}$$

Therefore,

$$\lim_{\nu\to 0}\Big(\mathcal{C}_q+\mathcal{D}_q\Big) = \frac{\sqrt{\pi}e^{\frac{i\pi}{4}}}{2(2\pi)^{\frac{3}{2}}\sqrt{H}} \tag{6.64}$$

We see that the coefficients are q-independent. Substituting eqs. (6.59) and (6.64) into eq. (6.55), we have the loop correlation function as

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x'})}\Big\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle_{loop,m=\frac{3H}{2}} = -\frac{27(8\pi GH^2)^2}{1024q^3} \tag{6.65}$$

which is exactly scale invariant.

6.5.2 Large Mass: $m > \frac{3H}{2}$

For $m > \frac{3H}{2}$, $\lambda_\pm$ are complex conjugate of each other. We define $\lambda_+ \equiv \lambda = -\frac{3}{2}+ir$, $\lambda_- = \lambda^* = -\frac{3}{2}-ir$ where $\nu = ir$ and $r \equiv \sqrt{|\frac{m^2}{H^2}-\frac{9}{4}|}$. Therefore, eq. (6.49) becomes

$$\pi(t)|_{m>\frac{3H}{2}} = \frac{1}{a^3}\Big((2\lambda^2H^2-m^2)C_pC_{p'}a^{2ir}+(2\lambda^{*2}H^2-m^2)D_pD_{p'}a^{-2ir} \tag{6.66}$$

$$+(2|\lambda|^2H^2-m^2)(C_pD_{p'}+D_pC_{p'})\Big) \tag{6.67}$$

$$\equiv a^{-3}\Big(c_+a^{2ir}+c_-a^{-2ir}+c_0\Big) \tag{6.68}$$

We see that the time component of the first two terms are complex conjugate of each other with different constant coefficients $c_\pm$. The last term approaches a^{-3} exactly. The factor a^{-3} in eq. (6.66) cancels with the factor $\sqrt{-g}$ in each interaction

Hamiltonian. Therefore, we expect that there is no divergence in the time integrals at late time when $t \to \infty$. The integral over time t_1 gives

$$\int_{-\infty}^{t_2} dt_1 e^{-iq\tau_1}\Big(c_+ a_1^{2ir} + c_- a_1^{-2ir} + c_0\Big) \quad \simeq \tag{6.69}$$

$$\frac{1}{2irH}\Big(c_+ e^{2irHt_2} - c_- e^{-2irHt_2} - 2irc_0 Ei(-iq\tau_2)\Big) \tag{6.70}$$

We see that there is an additional factor of $r^{-1} = \frac{H}{m}$ as a consequence of the time integration. The integral over time t_2 gives

$$\begin{aligned} \mathcal{T} &= -\frac{1}{rH} Re \int_{-\infty}^{t} dt_2 \sin q\tau_2 \Big(c_+^* e^{-2irHt_2} + c_-^* e^{2irHt_2} + c_0^*\Big) && (6.71)\\ &\quad \times\Big(c_+ e^{2irHt_2} - c_- e^{-2irHt_2} - 2irc_0 Ei(-iq\tau_2)\Big) && (6.72)\\ &= \frac{1}{rH^2}\int_{-\infty}^{0} \frac{d\tau_2}{\tau_2} \sin q\tau_2 \Big(|c_+|^2 - |c_-|^2 - 2r|c_0|^2 Siq\tau_2\Big) && (6.73)\\ &= \frac{1}{rH^2}\Big(\frac{\pi}{2}(|c_+|^2 - |c_-|^2) + \frac{r\pi^2}{4}|c_0|^2\Big) && (6.74) \end{aligned}$$

where the other terms vanish in the limit $t \to \infty$ because of the oscillating behavior of the integrand $e^{\pm iHt_2}$. We now need to calculate what $|c_{\pm,0}|^2$ are. From eq. (6.66), we have

$$\begin{aligned} |c_+|^2 &= |2\lambda^2 H^2 - m^2|^2 |C_p|^2 |C_{p'}|^2 && (6.75)\\ |c_-|^2 &= |2\lambda^2 H^2 - m^2|^2 |D_p|^2 |D_{p'}|^2 && (6.76)\\ |c_0|^2 &= (2|\lambda|^2 H^2 - m^2)^2 |C_p D_{p'} + D_p C_{p'}|^2 && (6.77) \end{aligned}$$

We can see from eq. (6.16) that $|C_q|^2$ and $|D_q|^2$ are *momentum independent* because $\nu \equiv ir$ is purely imaginary in the large mass limit when $m > \frac{3H}{2}$. Therefore,

$$|c_+|^2 - |c_-|^2 = \frac{\pi^2 |2\lambda^2 H^2 - m^2|^2 (e^{-2\pi r} - e^{2\pi r})}{8(2\pi)^6 H^2 |\sin \nu\pi|^4 |\Gamma(1+\nu)|^4} \tag{6.78}$$

Using $|\Gamma(1+ir)|^2 = |\Gamma(1-ir)|^2 = \frac{\pi r}{\sinh \pi r}$ and $\sin ir = i \sinh r$ for real r, the equation above is simplified as

$$|c_+|^2 - |c_-|^2 = -\frac{|2\lambda^2 H^2 - m^2|^2 \coth \pi r}{4(2\pi)^6 r^2 H^2} \tag{6.79}$$

where we use $\sinh 2x = 2 \sinh x \cosh x$. Also,

$$|c_0|^2 = \frac{(2|\lambda|^2 H^2 - m^2)^2 \cos^2(r \ln \frac{p}{p'})}{4(2\pi)^6 r^2 H^2 \sinh^2 \pi r} \tag{6.80}$$

Notice that the coefficients $|c_+|^2 \pm |c_-|^2$ are completely momentum independent and $|c_0|^2$ is nearly momentum independent ($\ln \frac{|\mathbf{p}|}{|\mathbf{p}+\mathbf{q}|} \to 0$ when $q \to 0$). Substituting eqs. (6.71), (6.79), and (6.80) into eq. (6.51), we have the loop correlation function due to massive scalar loop as

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x'})} \Big\langle \zeta(\mathbf{x}, t)\zeta(\mathbf{x}', t) \Big\rangle_{loop} = -\frac{2(2\pi)^3(8\pi G H^2(t_q))^2}{r H^2 q^6} \int d^3p d^3p' \tag{6.81}$$

$$\times \delta^3(\mathbf{q}+\mathbf{p}+\mathbf{p}') \Big(\frac{\pi}{2}(|c_+|^2 - |c_-|^2) + \frac{r\pi^2}{4}|c_0|^2 \Big) \tag{6.82}$$

Note that, in general,

$$\int d^3p \int d^3p' \delta^3(\mathbf{p}+\mathbf{p}'+\mathbf{q}) f = \frac{2\pi}{q} \int_0^\infty p dp \int_{|p-q|}^{p+q} p' dp' f \tag{6.83}$$

and

$$\int_{|p-q|}^{p+q} p' dp' f(p, p', q) = F(p+q) - F(|p-q|) \tag{6.84}$$

$$= 2q \frac{\partial F}{\partial p} + \mathcal{O}(q^3) \simeq 2pq f(p, p' = p, q) \tag{6.85}$$

We cut off the momentum integral over p at $\Lambda q \simeq q$ so that the approximation of the scalar mode solutions at low momentum in eq. (6.14) are still valid. Therefore, the result of the momentum integrals p, p' gives the momentum dependence as q^3, which cancels with the q^{-6} factor in eq. (6.81). We therefore have the approximated scale invariant spectrum as

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x'})}\Big\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle = \frac{\pi^2(8\pi G)^2}{3(2\pi)^3 r^3 q^3}|2\lambda^2H^2 - m^2|^2 \coth \pi r \tag{6.86}$$

where $\lambda = -\frac{3}{2} + ir$ and $r = \sqrt{|\frac{m^2}{H^2} - \frac{9}{4}|}$

We can see that, in the large m limit i.e when $m = M_{Pl}$ and $r \simeq \frac{M_{Pl}}{H}$, eq. (6.86) approaches

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x'})}\Big\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle_{m=M_{Pl}} \to \frac{\pi^2(8\pi G)^2 M_{Pl} H^3}{3(2\pi)^3 q^3} \tag{6.87}$$

where we use $(\sinh \pi r)^{-1} \to 0$ and $\coth \pi r \to 1$ in the large r limit.

We can compare the loop result when $m = M_{Pl}$ with the classical result and with the loop result when $m = 0$[2]. We have

$$\frac{\langle\zeta\zeta\rangle_{loop,m=M_{Pl}}}{\langle\zeta\zeta\rangle_{classical}} = \frac{4\pi^2|\epsilon|H(t_q)}{3M_{Pl}} \tag{6.88}$$

and

$$\frac{\langle\zeta\zeta\rangle_{loop,m=M_{Pl}}}{\langle\zeta\zeta\rangle_{loop,m=0}} = -\frac{5\pi M_{Pl}}{H \ln q} \tag{6.89}$$

We see that the spectrum is scale invariant. Also, it is smaller than the classical result by a factor of $\frac{H}{M_{Pl}}$ but larger than the massless minimal coupled scalar loop result calculated in [2] a by factor of $\frac{M_{Pl}}{H}$.

Chapter 7

Two Loops Density Perturbation

In the seen there will be only the seen;
in the heard there will be only the heard;
in the smelled, tasted, touched there will be
only the smelled, tasted, touched;
In the cognized there will be only the cognized. Gotama (Udana, I.x)

7.1 Why Two Loops?

In the previous chapters, we have studied the one-loop effect to the density perturbation during inflation. We found that the results are smaller than the classical result by a factor of $(\frac{H}{M_{Pl}})^2$ if the fields[1] circulated inside the loop are massless and by a factor of $\frac{H}{M_{Pl}}$ if the scalar field has a coupling and a mass as large as M_{Pl}(i.e. $\mathcal{L}_1 = \sqrt{-g}\varphi^2\sigma^2 = a^3(1+\frac{\dot{\zeta}}{H}+3\zeta)\bar{\varphi}^2\sigma^2$ for which $\bar{\varphi} \sim M_{Pl}$). Also, the one-loop results are all nearly scale invariance for all type of matter. It is natural to ask whether the

[1]We means *all* matter fields such as minimal coupled scalar calculated in [2], conformal scalar, fermion, and gauge fields calculated in previous chapters.

spectrum is still scale invariance and is in the order of the observed value beyond one-loop order. Since matter loop dominates when compared to gravity ζ loop, it is likely that the matter loop with a large coupling such as $\mathcal{L}_2 = \sqrt{-g}\varphi\sigma^3$ when $\bar{\varphi} \sim M_{Pl}$ may contribute to the power spectrum in the order of the classical result. However, after metric expansion, $\mathcal{L}_2$ can only contribute beyond one-loop power spectrum $\langle\zeta\zeta\rangle$. We therefore would like to calculate two-loop two point function of density perturbation in this chapter.

7.2 Large or Small?

As mentioned earlier, there is the possibility of large loop spectrum due to the large coupling in the order of M_{Pl}. However, we also need to examine the various commutators and the results of the integrals because these may be small enough to compensate the large vertices.

To obtain a quick estimation, we first consider the late time behavior that can arise in various commutators. Although there is an explicit factor of $\sqrt{-\bar{g}} = a^3$ in the interaction Hamiltonian, this is compensated by an unequal time commutator, which goes at most as a^{-3}. Since there are many commutators as well as many interactions, the largest result of N time integrals is in the order of $(\ln a)^N$[3]. With two-loop two point function considered here, there are two external legs of ζ_q which give a factor of $|\zeta_q^o|^4 \simeq \frac{H^4}{M_{Pl}^4\epsilon^2 q^6}$, two time dependent vertices (not include the field operators) which give a factor of $M_{Pl}^2 a_1^3 a_2^3$, and two unequal time commutators which give a factor of $a_1^{-3}a_2^{-3}$ at most for either ζ or σ fields. The rests are either ζ or σ fields that are not in the commutators and are all constant at late time. Therefore, the two-loop power spectrum will go at most as

$$\langle\zeta\zeta\rangle_{2-loop} \leq \frac{\bar{\varphi}(t_q)^2 H^4 \mathcal{C}_q}{M_{Pl}^4\epsilon^2 q^6}\int dt_2 \int dt_1 \rightarrow \frac{8\pi G H^2 \mathcal{C}_q \ln^2\tau}{\epsilon^2 q^6} \tag{7.1}$$

where $\mathcal{C}_q$ is the result of the three momentum integrals $\mathcal{C}_q = \int d^3p \int d^3p' \int d^3p'' f(p, p', p'', q)$. Since all fields in the propagators are massless, $\mathcal{C}_q$ would never be the function of M_{Pl}, unlike massive cases considered in the previous chapters. Therefore, it is possible that the order of magnitude of two-loop spectrum is larger or comparable to that of the classical result $\langle\zeta\zeta\rangle_{classical} \simeq 8\pi G H^2/\epsilon q^3$.

7.3 Two-Loop Two Point Function: The Momentum Dependence

We have estimated the upper bound of two-loop two point function in the previous sections and seen that the result will never be proportional to the positive power of a but may be roughly in the order of classical result. This upper bound is still larger than what is expected in perturbative quantization. We therefore like to see the actual result from a more rigorous calculation by expanding the full action and integrating it over times and momentum integrals. The action we considered is the usual Einstein gravity, inflaton with an arbitrary potential $V(\varphi)$, and additional massless minimal coupled scalar with unbroken symmetry $\langle\sigma\rangle = 0$ with coupling $\mathcal{L}_2$. The action considered here is the same as that in [2], except that we now allow additional interaction $\mathcal{L}_2$ that contributes a higher loop. For example, by expanding the metric $\sqrt{-g} = a^3(1 + \frac{\dot{\zeta}}{H})e^{3\zeta}$, we have the time dependent interaction vertices to fourth order of the fluctuated fields as

$$H_{\zeta\sigma\sigma\sigma}(t) = -\bar{\varphi}(t_q)a^3(t) \int d^3x \Big(\frac{\dot{\zeta}}{H} + 3\zeta\Big)\sigma^3 \tag{7.2}$$

From eq. (2.2), the correlation function at time t when $N = 2$ is

$$\Big\langle Q(t) \Big\rangle = -\int_{-\infty}^{t} dt_2 \int_{-\infty}^{t_2} dt_1 \Big\langle [H_1, [H_2, Q]] \Big\rangle_0 \tag{7.3}$$

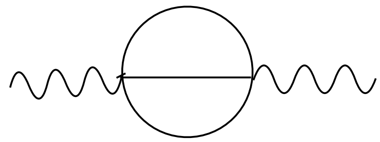

Figure 7.1: Two loop sunset diagram: Note that this graph includes all possible connections of $V_L(t)$ and $V_R(t)$ vertices with various $\Delta^{RR}(t_1, t_2)$ and $\Delta^{RL}(t_1, t_2)$ propagators. ζs are two external legs and σs are circulated inside the loops

where the vacuum expectation value in the RHS is in the free field vacuum and that in the LHS is in interacting vacuum. With the interaction in eq. (7.2), the equation above corresponds to the combination of $R - R$ and $R - L$ diagrams with two external ζ legs and two σ loops. The first commutator gives

$$\Big[H_2, Q\Big] = \bar{\varphi}(t_q) a_2^3(t) \int d^3x_2 \Big(\sigma_2^3 X_2 Q - Q\sigma_2^3 X_2\Big) \tag{7.4}$$

where

$$X \equiv \frac{\dot{\zeta}}{H} + 3\zeta \tag{7.5}$$

The second commutator gives

$$\Big[H_1, \Big[H_2, Q\Big]\Big] = \bar{\varphi}(t_q)^2 \int d^3x_1 \int d^3x_2 a_1^3 a_2^3 \Big[\sigma_1^3 X_1, \Big(\sigma_2^3 X_2 Q - Q\sigma_2^3 X_2\Big)\Big] \tag{7.6}$$

The interactions (7.2) contributes to two-loop two point function because they have one ζ and three σ lines. Since the vacuum expectation values of σ and ζ fields can be evaluated independently,

$$\Big\langle Q(t)\Big\rangle_{two-loop} = -\bar{\varphi}(t_q)^2 \int_{-\infty}^{t} dt_2 \int_{-\infty}^{t_2} dt_1 \int d^3x_1 \int d^3x_2 a_1^3 a_2^3 \Big(\langle\sigma_1^3\sigma_2^3\rangle_0 \tag{7.7}$$

$$(\langle X_1 X_2 Q\rangle_0 - \langle X_1 Q X_2\rangle_0) + \langle\sigma_2^3\sigma_1^3\rangle_0(\langle Q X_2 X_1\rangle_0 - \langle X_2 Q X_1\rangle_0)\Big) \tag{7.8}$$

where $Q(t) = \zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)$ is the product of field operators that we calculated. Since

$$\langle X_1X_2Q\rangle_0 = \langle QX_2X_1\rangle_0^*, \langle X_1QX_2\rangle_0 = \langle X_2QX_1\rangle_0^*, \langle\sigma_1^3\sigma_2^3\rangle_0 = \langle\sigma_2^3\sigma_1^3\rangle_0^* \tag{7.9}$$

we get

$$\Big\langle Q(t)\Big\rangle_{2-loops} = -2\bar{\varphi}(t_q)^2\int_{-\infty}^{t}dt_2\int_{-\infty}^{t_2}dt_1\int d^3x_1\int d^3x_2a_1^3a_2^3 \tag{7.10}$$

$$\times Re\Big(\langle\sigma_1^3\sigma_2^3\rangle_0(\langle X_1X_2Q\rangle_0 - \langle X_1QX_2\rangle_0)\Big) \tag{7.11}$$

To calculate the expectation values of various products of the field operators, we write σ and ζ in terms of creation and annihilation operators as

$$\sigma(\mathbf{x},t) = \int d^3p\Big(\alpha_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}\sigma_p(t) + \alpha_{\mathbf{p}}^*e^{-i\mathbf{p}\cdot\mathbf{x}}\sigma_p^*(t)\Big) \tag{7.12}$$

$$\zeta(\mathbf{x},t) = \int d^3p\Big(\beta_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}\zeta_p(t) + \beta_{\mathbf{p}}^*e^{-i\mathbf{p}\cdot\mathbf{x}}\zeta_p^*(t)\Big) \tag{7.13}$$

Therefore,

$$\langle\sigma_1^3\sigma_2^3\rangle_0 = 6\int\int\int d^3pd^3p'd^3p''e^{i(\mathbf{p}+\mathbf{p}'+\mathbf{p}'')\cdot(\mathbf{x}_1-\mathbf{x}_2)} \tag{7.14}$$

$$\sigma_p(t_1)\sigma_{p'}(t_1)\sigma_{p''}(t_1)\sigma_p^*(t_2)\sigma_{p'}^*(t_2)\sigma_{p''}^*(t_2) \tag{7.15}$$

where we normal order the product of field operators and use the commutation relations. The vacuum expectation value of normal ordering fields is zero. What are left are the product of three delta functions arisen from pairing six fields at different times in unequal time commutators(two fields give one delta function). The product of six field operators give six momentum integrals but the three delta functions eliminate three momentum integrals. Therefore, three momentum integrals are left as in eq. (7.14). For the ζ part,

$$\langle X_1 X_2 Q\rangle_0 - \langle X_1 Q X_2\rangle_0 = 2\int d^3k d^3k' e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x})+i\mathbf{k}'\cdot(\mathbf{x}_2-\mathbf{x}')} \tag{7.16}$$

$$X_k(t_1)\zeta_k^*(t)\Big(X_{k'}(t_2)\zeta_{k'}^*(t) - \zeta_{k'}(t)X_{k'}^*(t_2)\Big) \tag{7.17}$$

where $X_q \equiv \frac{\dot{\zeta}_q}{H} + 3\zeta_q$. Substituting eqs. (7.14), and (7.16) into eq. (7.10) and integrating over x_1, x_2 and x give

$$\int d^3x_1 \quad \rightarrow \quad (2\pi)^3\delta^3(\mathbf{k}+\mathbf{p}+\mathbf{p}'+\mathbf{p}'') \tag{7.18}$$

$$\int d^3x_2 \quad \rightarrow \quad (2\pi)^3\delta^3(-\mathbf{k}'+\mathbf{p}+\mathbf{p}'+\mathbf{p}'') \tag{7.19}$$

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')} \quad \rightarrow \quad (2\pi)^3\delta^3(\mathbf{q}-\mathbf{k}) \tag{7.20}$$

Hence,

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}\Big\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle_{two-loop} = -24(2\pi)^9\bar{\varphi}(t_q)^2 \tag{7.21}$$

$$\times Re\int_{-\infty}^{t} dt_2 \int_{-\infty}^{t_2} dt_1 a_1^3 a_2^3 \int d^3p \int d^3p' \int d^3p'' \delta^3(\mathbf{q}+\mathbf{p}+\mathbf{p}'+\mathbf{p}'') \tag{7.22}$$

$$\times\sigma_p(t_1)\sigma_{p'}(t_1)\sigma_{p''}(t_1)\sigma_p^*(t_2)\sigma_{p'}^*(t_2)\sigma_{p''}^*(t_2)\Big[\frac{\dot{\zeta}_q(t_1)}{H} + 3\zeta_q(t_1)\Big]\zeta_q^*(t) \tag{7.23}$$

$$\times\Big[\Big(\frac{\dot{\zeta}_q(t_2)}{H} + 3\zeta_q(t_2)\Big)\zeta_q^*(t) - \zeta_q(t)\Big(\frac{\dot{\zeta}_q^*(t_2)}{H} + 3\zeta_q^*(t_2)\Big)\Big] \tag{7.24}$$

With the mode solutions of ζ_q and σ_p in de-Sitter space,

$$\zeta_q(t) = \zeta_q^o e^{-iq\tau}(1+iq\tau), \dot{\zeta}_q(t) = -\frac{\zeta_q^o q^2}{Ha^2}e^{-iq\tau} \tag{7.25}$$

$$\sigma_q(t) = \sigma_q^o e^{-iq\tau}(1+iq\tau) \tag{7.26}$$

where

$$|\sigma_q^o|^2 = \frac{H^2(t_q)}{2(2\pi)^3 q^3} \tag{7.27}$$

$$|\zeta_q^o|^2 = \frac{8\pi G H^2(t_q)}{2(2\pi)^3 \epsilon(t_q) q^3} \tag{7.28}$$

we have

$$\frac{\dot{\zeta}_q(t)}{H} + 3\zeta_q(t) = \zeta_q^o e^{-iq\tau}\Big(3 + 3iq\tau - q^2\tau^2\Big) \equiv \zeta_q^o Y_q(t) \tag{7.29}$$

and

$$\sigma_p(t)\sigma_{p'}(t)\sigma_{p''}(t) = \sigma_p^o\sigma_{p'}^o\sigma_{p''}^o e^{-i(p+p'+p'')\tau}\Big(1 + i(p+p'+p'')\tau \tag{7.30}$$

$$-(pp' + pp'' + p'p'')\tau^2 - ipp'p''\tau^3\Big) \equiv \sigma_p^o\sigma_{p'}^o\sigma_{p''}^o S(t) \tag{7.31}$$

Hence, eq. (7.21) can be written in a more compact form as

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}\Big\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle_{2-loops} = -24(2\pi)^9|\zeta_q^o|^4\bar{\varphi}(t_q)^2 \tag{7.32}$$

$$\int d^3p\int d^3p'\int d^3p''\delta^3(\mathbf{q}+\mathbf{p}+\mathbf{p}'+\mathbf{p}'')|\sigma_p^o|^2|\sigma_{p'}^o|^2|\sigma_{p''}^o|^2 \tag{7.33}$$

$$Re\int_{-\infty}^0 d\tau_2\int_{-\infty}^{\tau_2} d\tau_1 a_1^4 a_2^4 S^*(t_2)S(t_1)Y_q(t_1)\Big(Y_q(t_2) - Y_q^*(t_2)\Big) \tag{7.34}$$

where we first integrate over times before spatial momentums and take the upper limit $t \to \infty$ or $\tau \to 0$ for sufficiently late time during inflation. From eqs. (7.29) and (7.30), the time integral over t_1 is

$$\int_{-\infty}^{\tau_2} d\tau_1 a_1^4 S(t_1)Y_q(t_1) \tag{7.35}$$

$$= \frac{1}{H^4}\int_{-\infty}^{\tau_2}\frac{d\tau_1}{\tau_1^4}e^{-ik_t\tau_1}[3 + 3iq\tau_1 - q^2\tau_1^2][1 + ir\tau_1 - u\tau_1^2 - iv\tau_1^3] \tag{7.36}$$

where

$$
\begin{aligned}
k_t &\equiv p + p' + p'' + q & (7.37)\\
r &\equiv p + p' + p'' = k_t - q & (7.38)\\
u &\equiv pp' + pp'' + p'p'' & (7.39)\\
v &\equiv pp'p'' & (7.40)
\end{aligned}
$$

The result of the first time integral is

$$
\begin{aligned}
\mathcal{T}_2 &\equiv \int_{-\infty}^{t_2} dt_1 a_1^3 S(t_1) Y_q(t_1) & (7.41)\\
&= \frac{e^{-ik_t\tau_2}}{H^4}\Big[-\frac{1}{\tau_2^3} - \frac{ik_t}{\tau_2^2} + \frac{1}{\tau_2}\Big(-k_t^2 + 3qk_t - 2q^2 + 3u\Big) + & (7.42)\\
&\quad \frac{iq}{k_t^2}\Big(3k_t v + qk_t u + qv\Big) - \frac{q^2 v\tau_2}{k_t} + ie^{ik_t\tau_2} Ei(-ik_t\tau_2) & (7.43)\\
&\quad \times\Big(-k_t^3 + 3qk_t^2 + 3(u - q^2)k_t + (q^3 - 3qu - 3v)\Big)\Big] & (7.44)
\end{aligned}
$$

where we rotate the integral contour $\tau \to i\tau$ when $\tau \to -\infty$. Therefore, the time integral converges at early time due to all fluctuations oscillating very rapidly as $e^{-ik_t\tau}$. From eqs. (7.29) and (7.30), we have

$$
S_2^*\Big(Y_2 - Y_2^*\Big) = e^{ik_t\tau_2}\Big(1 - ir\tau_2 - u\tau_2^2 + iv\tau_2^3\Big)\Big(e^{-2iq\tau_2}h - h^*\Big) \quad (7.45)
$$

where $h \equiv 3 + 3iq\tau_2 - q^2\tau_2^2$. As a result the integral over time t_2 is

$$\mathcal{T} \equiv Re\int_{-\infty}^{0} d\tau_2 a_2^4 S^*(t_2)\Big(Y_q(t_2) - Y_q^*(t_2)\Big)\int_{-\infty}^{\tau_2} d\tau_1 a_1^4 S(t_1) Y_q(t_1) \quad (7.46)$$

$$= \frac{1}{H^8} Re\int_{-\infty}^{0}\frac{d\tau_2}{\tau_2^4}\Big(1 - ir\tau_2 - u\tau_2^2 + iv\tau_2^3\Big)\Big(e^{-2iq\tau_2}h - h^*\Big) \quad (7.47)$$

$$\Big[-\frac{1}{\tau_2^3} - \frac{ik_t}{\tau_2^2} + \frac{1}{\tau_2}\Big(-k_t^2 + 3qk_t - 2q^2 + 3u\Big) + \quad (7.48)$$

$$\frac{iq}{k_t^2}\Big(3k_t v + qk_t u + qv\Big) - \frac{q^2 v\tau_2}{k_t} + ie^{ik_t\tau_2}Ei(-ik_t\tau_2) \quad (7.49)$$

$$\times\Big(-k_t^3 + 3qk_t^2 + 3(u - q^2)k_t + (q^3 - 3qu - 3v)\Big)\Big] \quad (7.50)$$

We can quickly check that there is no infrared divergence when $\tau \to 0$ in the integral above by expanding $e^{-2iq\tau_2}h - h^*$ in the integrand. This gives $e^{-2iq\tau_2}h - h^* \to \mathcal{O}(i\tau^5)$ where the leading order in τ is subsequently cancelled by the factors $\frac{1}{\tau^7}$ and $\frac{i}{\tau^6}$ in the integrand. The first integral gives $\int^0 \frac{i\tau^5}{\tau^7} \to \frac{i}{0}$ but this result is purely imaginary. Therefore there is no contribution from the first integral after taking the real part. The second integral, which is real, gives $\int^0 \frac{\tau^5}{\tau^6} \to \ln 0$. This result is roughly equivalent to $\ln a(t)$ when t is sufficiently late during inflation $t \to \infty$.

The prior analysis is only used for convergence check. It is necessary to evaluate the integrals *before* taking the upper limit $\tau \to 0$. We use Mathematica to evaluate the integral above and obtain the result in order of polynomial in q as

$$\mathcal{T} = -\frac{q}{36k_t^2}\Big[\Big((108\gamma_E - 198)r^4 v + (594 - 324\gamma_E)r^2 uv \quad (7.51)$$

$$+(324\gamma_E - 279)rv^2\Big) + q\Big((36\gamma_E - 108)r^4 u + (324 - 108\gamma_E)r^2u^2 \quad (7.52)$$

$$+(144\gamma_E - 306)r^3 v + (594 - 324\gamma_E)ruv + (432\gamma_E - 342)v^2\Big) \quad (7.53)$$

$$+q^2\Big(-12r^5 + (36\gamma_E - 84)r^3 u + (450 - 108\gamma_E)ru^2 - 240r^2 v \quad (7.54)$$

$$+(108\gamma_E - 198)uv\Big) + q^3\Big(12r^2u + 126u^2 - 264rv\Big) \quad (7.55)$$

$$+q^4(36r^3 - 60ru - 36v) + q^5(28r^2 - 48u) + 8rq^6 + 4q^7\Big] \quad (7.56)$$

where $\gamma_E = \gamma + \ln 2 \simeq 1.26$. The function $Ei(x)$ in the t_2 integral gives a result in the order of $\mathcal{O}(\tau^2)$ that approaches zero when we take the upper limit $\tau \to 0$[2]. Notice that the result of the time integrals above in eq. (7.51) is exact and has *no* singularity in q, unlike the result in one-loop which contains an additional factor q^{-1}. The only singularity in q when $q \to 0$ comes from the factor $|\zeta_q^o|^4 \to q^{-6}$ but *not* from the result of the time integral. Substituting eqs. (7.27) and (7.51) into eq. (7.32), we have

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')} \Big\langle \zeta(\mathbf{x},t)\zeta(\mathbf{x}',t) \Big\rangle_{2-loops} = -\frac{3(8\pi G H(t_q)\bar{\varphi}(t_q))^2}{4(2\pi)^6\epsilon^2 q^6} \tag{7.57}$$

$$\times \int d^3p \int d^3p' \int d^3p'' \delta^3(\mathbf{q}+\mathbf{p}+\mathbf{p}'+\mathbf{p}'') f(p,p',p'',q) \tag{7.58}$$

where

$$f(p,p',p'',q) \quad = \quad \frac{\mathcal{T}}{p^3 p'^3 p''^3} \tag{7.59}$$

By power counting, we see that the momentum integral shown above is *quadratic* UV divergence rather than the quartic divergence in one-loop. For two-loop, quadratic divergence is the highest divergence we could have because higher polynomial terms in q could only give sub-divergence. Therefore, the momentum dependence in the final result *never* be far from scale invariance and will not change even in higher polynomial q-terms in eq. (7.51) because the UV divergences are of higher polynomial q-terms are less than quadratic. We still keep exact result in eq. (7.51) to calculate a precise coefficient in the next section. With dimension regularization, we can write the momentum integrals as

$$\int d^3p \int d^3p' \int d^3p'' \delta^3(\mathbf{q}+\mathbf{p}+\mathbf{p}'+\mathbf{p}'') f(p,p',p'',q) \Rightarrow q^{2+\delta} F(\delta) \tag{7.60}$$

[2]This means the time that is still during inflation but sufficiently late i.e. with sufficient numbers of e-folding.

where δ is a measure of the difference between the space dimensionality and three. The UV divergence for $\delta = 0$ gives

$$F(\delta) \to F_0 + \frac{F_1}{\delta} + \frac{F_2}{\delta^2} \tag{7.61}$$

So that in the limit $\delta = 0$, we expand $q^\delta = e^{\delta \ln q} = 1 + \delta \ln q + \frac{\delta^2 \ln^2 q}{2} + \mathcal{O}(\delta^3)$

$$\int d^3p \int d^3p' \int d^3p'' \delta^3(\mathbf{q} + \mathbf{p} + \mathbf{p}' + \mathbf{p}'') f(p, p', p'', q) \tag{7.62}$$

$$= q^2 \Big[\frac{F_2}{2} \ln^2 q + F_1 \ln q + F_0 + L \Big] \tag{7.63}$$

where L is a divergent constant and $F_{0,1,2}$ are the coefficient numbers determined from the actual integrals. Therefore, we still have the nearly scale invariant spectrum at two loop level as

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')} \Big\langle \zeta(\mathbf{x}, t)\zeta(\mathbf{x}', t) \Big\rangle_{2-loop} \tag{7.64}$$

$$= \quad -\frac{3(8\pi G)^2 \bar{\varphi}^2(t_q) H^2(t_q)}{4(2\pi)^6 \epsilon^2 q^3} \Big[\frac{F_2}{2} \ln^2 q + F_1 \ln q + F_0 + L \Big] \tag{7.65}$$

We see that the order of magnitude of this result can be in the order of classical result $(8\pi G H^2(t_q))$ if $\bar{\varphi}(t_q) \sim M_{Pl}$ at the time of horizon exit. We obtain the factor $\ln^2 q$ because the two loop effect gives a pole term in the order of $\mathcal{O}(\delta^{-2})$ when dimensional regularization is used.

7.4 Can the departure from scale invariance be large?: Calculate the coefficient

It is important to have some idea whether a constant coefficient F_2 is large or small. The reason is that the number of $8\pi G$ at the two-loop quantum level can be the same as that at the classical level if $\bar{\varphi}(t_q)$ is as large as M_{Pl}. If the numerical coefficient to leading order of $\ln q$ (which is $\ln^2 q$ for two-loop here) is found to be too large (i.e 10^5), we can conclude that the theories considered are invalid because we never can observe anything much larger than the classical result. If the coefficient is not far greater than the observed value, it is most likely possible that what we are doing is in the right direction. Therefore, we would like to calculate the coefficient to see whether the departure from scale invariance can be large.

To calculate the numerical coefficient, we need to do the actual momentum integrals and differentiate with respect to external momentum q to lower the degree of divergence until we obtain finite integrals. Note that in general

$$\mathcal{I}(q) \equiv \int d^3p \int d^3p' \int d^3p'' \delta^3(\mathbf{p}+\mathbf{p}'+\mathbf{p}''+\mathbf{q}) f(p,p',p'',q) \tag{7.66}$$

$$= \int d^3p \int d^3p' f(p,p',p''=|\mathbf{q}+\mathbf{p}+\mathbf{p}'|,q) \tag{7.67}$$

We can choose the direction of $\mathbf{q}$ in z direction because it is a fixed external momentum of ζ. Therefore,

$$\mathcal{I}(q) = 2\pi \int_0^\infty p^2 dp \int_0^\infty p'^2 dp' \int_{-1}^1 d(\cos\theta_{qp}) \int_{-1}^1 d(\cos\theta_{qp'}) \tag{7.68}$$

$$\times \int_0^{2\pi} d\varphi_{pp'} f(p,p',p''=|\mathbf{q}+\mathbf{p}+\mathbf{p}'|,q) \tag{7.69}$$

Since

$$p''^2 = p^2 + p'^2 + q^2 + 2qp\cos\theta_{qp} + 2qp'\cos\theta_{qp'} + 2pp'\cos\theta_{pp'} \tag{7.70}$$

we need to find the angle $\theta_{pp'}$ as a function of θ_{qp} and $\theta_{qp'}$. Therefore,

$$\cos\theta_{pp'} = \frac{\mathbf{p}\cdot\mathbf{p}'}{pp'} \tag{7.71}$$

$$= \sin\theta_{qp}\sin\theta_{qp'}\cos\varphi_{pp'} + \cos\theta_{qp}\cos\theta_{qp'} \tag{7.72}$$

where $\varphi_{pp'} = \varphi_{qp} - \varphi_{qp'}$. Define $\cos\theta_{qp} = x$ and $\cos\theta_{qp'} = y$, we have

$$\mathcal{I}(q) = 2\pi \int_0^\infty p^2 dp \int_0^\infty p'^2 dp' \int_{-1}^1 dx \int_{-1}^1 dy \int_0^{2\pi} d\varphi_{pp'} \frac{\mathcal{T}(p,p',p'',q)}{p^3 p'^3 p''^3} \tag{7.73}$$

We differentiate with respect to q three times to lower the degree of quadratic divergence. Note that the exact result of time integrals from the previous section is given in eq. (7.51)

From eq. (7.51), we see that the terms proportional to q, q^2 and q^3 give quadratic, linear, and logarithmic UV divergence respectively. The terms proportional to q^4 and higher polynomial in q are finite at large p, p'. However, the higher polynomial in q terms are dominated at low momentum p, p' and the result goes as $\ln p \ln p'$ when $p, p' \to 0$. This can be seen by simple dimension analysis. To calculate finite contribution of the spectrum, we differentiate f with respect to q three times. We would like to calculate the coefficient of the $\ln^2 q$. Therefore, we need to evaluate the convergent integrals of

$$\frac{d^3\mathcal{I}(q)}{dq^3} = 2\pi \int_0^\infty p^2 dp \int_0^\infty p'^2 dp' \int_{-1}^1 dx \int_{-1}^1 dy \int_0^{2\pi} d\varphi_{pp'} \frac{d^3 f(p,p',p'',q)}{dq^3} \tag{7.74}$$

where

$$p''^2 = p^2 + p'^2 + q^2 + 2qpx + 2qp'y + 2pp'(\sqrt{(1-x^2)(1-y^2)}\cos\varphi_{pp'} + xy) \tag{7.75}$$

Note that

$$w' \equiv \frac{dp''}{dq} = \frac{s}{p''} \tag{7.76}$$

$$w'' = \frac{d^2p''}{dq^2} = \frac{1}{p''}\left(1 - \frac{s^2}{p''^2}\right) \tag{7.77}$$

$$w''' = \frac{d^3p''}{dq^3} = -\frac{3s}{p''^3}\left(1 - \frac{s^2}{p''^2}\right) \tag{7.78}$$

where

$$s \equiv px + p'y + q \tag{7.79}$$

We are not able to do the momentum integrals eq. (7.74) analytically without an approximation because the function f in eq. (7.51) is too complicated. However, an approximated integrals can be done analytically by noticing that the integrals over momentums in eq. (7.74) converge when p and p' go to ∞. The reason is that $\frac{d^3 f}{dq^3}$ goes as p^{-7} when p' is in the same footing as in p. Therefore, the result of the integrals over p and p' converges rapidly as $\frac{1}{p} \to \frac{1}{\infty} \to 0$. We can therefore expect that the main contribution of the integral above come from the infrared regime in which $p \approx p' \leq q$. With p, p' are sufficiently small, $s, p'' \to q, k_t \to 2q, w' \to 1$, $r \to q$, $u \to 0$, $v \to 0$ and $w'', w''' \to 0$. We therefore have

$$\frac{d^3 f}{dq^3} \to \frac{1}{(pp'q)^3}\Big[- \frac{60\mathcal{T}}{q^3} + \frac{36}{q^2}\frac{d\mathcal{T}}{dq} - \frac{9}{q}\frac{d^2\mathcal{T}}{dq^2} + \frac{d^3\mathcal{T}}{dq^3}\Big] \tag{7.80}$$

where

$$\mathcal{T} \to -\frac{4q^6}{9} \tag{7.81}$$

$$\frac{d\mathcal{T}}{dq} \to -\frac{8q^5}{3} \tag{7.82}$$

$$\frac{d^2\mathcal{T}}{dq^2} \to -\frac{40q^4}{3} \tag{7.83}$$

$$\frac{d^3\mathcal{T}}{dq^3} \to -\frac{160q^3}{3} \tag{7.84}$$

Hence,

$$\frac{d^3 f}{dq^3} \to -\frac{8}{3(pp')^3} \tag{7.85}$$

As expected, we see that the main contribution of three derivative integral comes from infrared regime when $p, p' \ll q$. Substituting equation above back in eq. (7.74), we have

$$\frac{d^3\mathcal{I}(q)}{dq^3} = -\frac{128\pi^2}{3}\int_0^q dp \int_0^q dp' \frac{1}{pp'} \tag{7.86}$$

$$= -\frac{128\pi^2}{3}\Big[\ln^2 q - 2\ln 0 \ln q + \ln^2 0\Big] \tag{7.87}$$

Integrate over q back, we have

$$\mathcal{I}(q) = -\frac{64\pi^2 q^3}{9}\Big[\ln^2 q - 2\ln 0 \ln q + \ln^2 0\Big] + C \tag{7.88}$$

where C is a divergent constant. Substituting equation above back in eq.(7.57), we have two-loop spectrum as

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}\Big\langle \zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle_{2-loop} = \tag{7.89}$$

$$\frac{[8\pi G H(t_q)\bar{\varphi}(t_q)]^2}{12\pi^4\epsilon^2 q^3}\Big[\ln^2 q - 2\ln 0 \ln q + \ln^2 0 + L\Big] \tag{7.90}$$

To compare with the classical result

$$\int d^3x e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}\Big\langle \zeta(\mathbf{x},t)\zeta(\mathbf{x}',t)\Big\rangle_{classic} = \frac{8\pi G H^2(t_q)}{4(2\pi)^3|\epsilon(t_q)|q^3} \tag{7.91}$$

we have

$$\frac{\langle\zeta\zeta\rangle_{2-loop}}{\langle\zeta\zeta\rangle_{classic}} = \frac{8(8\pi G)\bar{\varphi}^2(t_q)}{3\pi|\epsilon|}\Big[\ln^2 q - 2\ln 0 \ln q + \ln^2 0 + L\Big] \tag{7.92}$$

We see that if $\bar{\varphi}(t_q)$ is as large as M_{Pl}, the two loop result can be in order of classical value but the momentum dependence never be far different from q^{-3} .

7.5 Comment on n-Loop

We have noticed that the momentum dependence of power spectrums when dimensional regularization is used goes as $\frac{F_1 \ln q + F_0}{q^3}$ and $\frac{F_2 \ln^2 q + F_1 \ln q + F_0}{q^3}$ for one and two loops respectively. We can therefore expect the momentum dependence of n loops two point function to be

$$\langle \zeta\zeta \rangle_{n-loops} \to \frac{\alpha^2 (8\pi G H^2)^2}{q^3} \sum_{k=0}^{n} F_k (\ln q)^k \tag{7.93}$$

where α is some constant and F_k are numerical coefficients. The factor $\sum_{k=0}^{n} F_k \ln^k q$ occurs because the pole terms go as $\sum_{k=1}^{n} \frac{1}{\delta^k}$ for n-loop. However, the interactions that generate $n > 2$ loops will *not* have a coupling in dimension of mass. This means that, unlike the two-loop result, the coupling cannot be as large as M_{Pl} to cancel the factor $8\pi G$. We can arrive at this conclusion through dimensional counting. For example, the interactions that contribute to a three-loop sunset diagram is $\mathcal{L}_{int} = a^3 \alpha \zeta \sigma^4$ in which α is dimensionless.

Chapter 8

Conclusion and Outlook

Be it little or much that you can tell,

The meaning only, please proclaim to me!

To know the meaning is my sole desire;

Of no use to me are many words.

Sariputta.

8.1 Summary of All Results

Quantum effect of cosmological correlations due to the interactions of gravitational and all matters fluctuations are studied. It is shown that the departure from scale invariance never be greater than order one (the momentum dependence goes as $q^{-3+\eta}$ such that $|\eta| < 1$, always), regardless what kind of theories, what kind of matters, or what kind of inflaton potential $V(\varphi)$ is. As described in appendix, the results are also valid to more general potential $V(\varphi, \sigma, \bar{\psi}\psi, A_\mu A^\mu)$ in which an inflaton additionally interacts with arbitrary kinds of matters such as scalar, fermion, gauge fields with and without mass. The results from each chapters are

Dirac Field Loops

$$\langle\zeta\zeta\rangle_{m=0} = -\frac{4\pi(8\pi G)^2 H(t_q)^4}{15(2\pi)^3 q^3}\Big[\ln q + L\Big] \tag{8.1}$$

where $H(t_q)$ is the expansion rate at the time of horizon exit and

$$\langle\zeta\zeta\rangle_{m\neq 0} \to -\frac{7(8\pi G)^2 m^2 H(t_q)^2}{24 q^3 \cosh^2\frac{\pi m}{H}} \tag{8.2}$$

Gauge Field Loops

$$\langle\zeta\zeta\rangle_{m=0} = -\frac{14\pi(8\pi G)^2 H(t_q)^4}{5(2\pi)^3 q^3}\Big[\ln q + L\Big] \tag{8.3}$$

$$\langle\zeta\zeta\rangle_{m<\frac{H}{2}} \to -\frac{24\Gamma^2(\nu)\lambda_-\lambda_+^3(8\pi G H(t_q)^2)^2}{(2\pi)^3[2H(t_q)a(t)]^\eta}\frac{(1-\eta\ln q\tau)}{(1-\eta)(2+\eta)\eta^2 q^{3-\eta}} \tag{8.4}$$

where $0 \ll 2\nu = \sqrt{1-\frac{4m^2}{H^2}} < 1$ and $\eta = 1-2\nu$.

$$\langle\zeta\zeta\rangle_{m=\frac{H}{2}} \quad = \quad -\frac{\pi^2(8\pi G)^2 H(t_q)^4}{1024 q^3} \tag{8.5}$$

$$\langle\zeta\zeta\rangle_{m>\frac{H}{2}} \to \frac{3\pi^2(8\pi G)^2 H(t_q)^4|\lambda|^4}{8r^3(2\pi)^3 q^3}\coth\pi r \tag{8.6}$$

where $\lambda = -\frac{1}{2} + ir$ and $r = \sqrt{|\frac{m^2}{H^2} - \frac{1}{4}|}$.

$$\langle\zeta\zeta\rangle_{m=M_{Pl}} \to \frac{3\pi^2(8\pi G)^2 M_{Pl} H(t_q)^3}{8(2\pi)^3 q^3} \tag{8.7}$$

Conformal Scalar Field Loops

$$\langle\zeta\zeta\rangle = -\frac{\pi(8\pi G)^2 H(t_q)^4}{90(2\pi)^3 q^3}\Big[\ln q + L\Big] \tag{8.8}$$

Minimal Coupled Scalar Field Loops

$$\langle\zeta\zeta\rangle_{m=0} = -\frac{\pi(8\pi G)^2 H(t_q)^4}{15(2\pi)^3 q^3}\Big[\ln q + L\Big] \tag{8.9}$$

Note that the result above found by Weinberg[2]

$$\langle\zeta\zeta\rangle_{m=\frac{3H}{2}} = -\frac{27(8\pi G)^2 H(t_q)^4}{1024 q^3} \tag{8.10}$$

and

$$\langle\zeta\zeta\rangle_{m>\frac{3H}{2}} \to = \frac{\pi^2(8\pi G)^2}{3(2\pi)^3 r^3 q^3}|2\lambda^2 H(t_q)^2 - m^2|^2 \coth \pi r \tag{8.11}$$

where $\lambda = -\frac{3}{2} + ir$ and $r = \sqrt{|\frac{m^2}{H^2} - \frac{9}{4}|}$

$$\langle\zeta\zeta\rangle_{m=M_{Pl}} \to \frac{\pi^2(8\pi G)^2 M_{Pl} H(t_q)^3}{3(2\pi)^3 q^3} \tag{8.12}$$

Minimal Coupled Scalar Field Two-Loop

$$\langle\zeta\zeta\rangle_{m=0} \to \frac{(8\pi G)^2 H(t_q)^2 \bar{\varphi}(t_q)^2}{12\pi^4\epsilon^2 q^3}\Big[\ln^2 q - 2\ln 0 \ln q + \ln^2 0 + L\Big] \tag{8.13}$$

Note that the result above is from a calculation of two-loop sunset diagram only.

General n-Loops

$$\langle\zeta\zeta\rangle_{n-loop} \to \frac{\alpha^2(8\pi G)^2 H(t_q)^4}{q^3}\sum_{k=0}^{n} F_k(\ln q)^k \tag{8.14}$$

where α is a constant arise from vertices and F_k is a constant resulting from the actual time and momentum integrals.

It is interesting to see that although different kind matters such as minimal coupled scalar, conformal scalar, fermion, and gauge fields appear to be different through their different wave functions in inflating universe and their different interactions with gravitational fluctuation in the actions, they all share the same nature in the observable correlation functions. They all predict nearly scale invariant spectrums that seed the large scale structure of the universe observed today. This implies that what are observed today such as fermion and vector fields are all also there as the quantum fluctuations in the past even during the time of Big Bang inflation.

8.2 An Outlook

We have studied the possibilities of matters other than inflaton during inflation. There is some learning that is worthwhile to be mentioned and investigated further.

1. Loop effect is generally suppressed by an additional factor of GH^2. However, if one take a coupling constant as large as M_{Pl}[1], one may believe that the result would *not* be suppressed by GH^2. Without GH^2 suppression, one may lead to the false conclusion that the quantum effect can be larger than or in the same order of magnitude as the classical value and hence perturbation theory breaks down. However such one-loop theories can only arise in massive but not massless theories because inflaton and gravity fluctuate around non-zero background and contribute to the effective mass term with $m \sim M_{Pl}$. This means that if interactions are as large as M_{Pl} propagators also acquires mass as large as M_{Pl}. Therefore more careful analysis is required. In this dissertation, we showed that the results for massive theories are still suppressed by the mass

[1]This is possible in realistic inflationary theories because of possible coupling with an unperturbed inflaton $\bar{\varphi} \sim M_{Pl}$

dependent terms that can arise through mass dependent propagators and loop integrations. Therefore cosmological perturbation theory is still valid at one-loop level even when the vertices are as large as M_{Pl}.

2. For two-loop effect, it is not necessary to have mass in the way as described earlier. Therefore, the two-loop quantum spectrum can be as large as the classical spectrum if the coupling is as large as M_{Pl}. However, the calculation showed that the coefficient of $\ln^2 q$ is still less than one.

3. The actual and physical spectrum $\langle\zeta\zeta\rangle$ is a "full" two-point correlation function with summation of all possible diagrams and all loops. We have only calculated a two-loop sunset diagram. As we noted in the n-loop section, the higher-loop also contributes a sum of $\ln^k q$. This means that part of the higher-loop contributes to the coefficient of $\ln^2 q$. Other topology of diagrams such as two-loop Master diagram also contribute similarly. Therefore, the combination of all the above will change all coefficients of $\ln^k q$.

4. As seen from the results in Chapters $3-7$, the momentum dependence of the spectrum goes as $q^{-3+\eta(m)}$ when the cut-off is used and is $\sum_k q^{-3}\ln^k q$ when the dimension regularization is used. This raises the question whether the momentum dependence of the spectrum may be regularization dependent. Although both these two methods of regularization agree on the nearly scale invariant result, it may be worth trying other ways of regularization such as Paulli-Villar, point-splitting or zeta-function regularization to investigate if the departure from scale invariance can be large.

5. The nearly scale invariant results of massive theories of scalar, fermion, and vector fields shown in this dissertation are also valid for a more general potential $V(\varphi,\sigma,\bar{\psi}\psi,A_\mu A^\mu)$ of arbitrary interactions of inflaton and all matters. We have worked with a $\delta\varphi=0$ gauge and therefore ensure that whatever

additional interaction that arises in the potential does not change the results in eqs. (8.2), (8.6), and (8.11) but only shifts the mass terms (as described in appendix). The reason is that an unperturbed inflaton does not change much during inflation and therefore those arbitrary interactions can only shift the mass in massive matters and gravitational theories. It will be interesting for one to further explore and find any example that might not give the same nearly scale invariant results for more general potential $V(\varphi, \sigma, \bar{\psi}\psi, A_\mu A^\mu)$ claimed in this dissertation.

Appendix
Gravity and General Matter Interactions

Whatever portion of the journey has been completed
give the inspiration and strength to walk on.
And the greatest help upon the path is gratitude.
This is the support for the journey ahead. S. N. Goenka

This appendix is to clarify and derive interactions of matter and gravitational fluctuations used in chapter 3-6. The method of expansion and quantization shown by Weinberg[2] has a more compact form than the direct expansion of matter and gravitational fluctuations. Following his method, we could extend the calculation to other matters such as fermion, gauge, and conformal scalar fields without much difficulties. We like to show the calculation in details for the general reader.

A.1 Higher-Order Fluctuations

In cosmological fluctuations, we generally expand the gravity and inflaton around time dependent background such that

$$g_{\mu\nu}(\mathbf{x},t) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(\mathbf{x},t) \tag{A-1}$$

$$\varphi(\mathbf{x},t) = \bar{\varphi}(t) + \delta\varphi(\mathbf{x},t) \tag{A-2}$$

When we add any other matters which have unbroken symmetries, they can be expanded as

$$M(\mathbf{x},t) = 0 + \delta M(\mathbf{x},t) \tag{A-3}$$

where M represents any additional matters such as fermion, gauge, and conformal scalar fields. The perturbation to the metric around FRW background can always be placed in the form of

$$ds^2 = -(1+E)dt^2 + 2a(t)F_{,i}dtdx^i + a^2(t)((1+A)\delta_{ij} + B_{,ij}) \tag{A-4}$$

where we only consider the scalar mode which is the subject of interest here. The gauge invariant observable quantity is defined as

$$\zeta_q \equiv \frac{A_q}{2} - \frac{H\delta\rho_q}{\dot{\rho}} \tag{A-5}$$

to the linear order. We see that it relates to both matter and gravitational fluctuations. There is a need for us to learn how to quantize such theories with minimum complication.

Since inflaton and gravity are related through the Einstein's equation, we have some choices in choosing a gauge. It is found to be more convenient to choose a gauge such that inflaton does not fluctuate ($\delta\varphi_q = 0$)[7] rather than a gauge that is on the gravity side. With a $\delta\varphi = B = 0$ gauge, the quantity ζ_q is purely gravity. Therefore, we can write down all the components of gravitational fluctuations $\delta g_{\mu\nu}$ in terms of a single variable ζ by solving Einstein's equation in Maldacena gauge $\delta\varphi = 0$. From the gravitational field equations and the energy conservation equations[2],

$$
\begin{aligned}
0 &= \dot{A} - HE && \text{(A-6)}\\
0 &= H\dot{E} + 2(3H^2 + \dot{H})E + a^{-2}\nabla^2 A - \ddot{A} - 6H\dot{A} + 2a^{-1}H\nabla^2 F && \text{(A-7)}\\
0 &= -\frac{1}{2}\frac{d}{dt}(E\dot{H}) - 3H\dot{H}E - a^{-1}\dot{H}\nabla^2 F + \frac{3}{2}\dot{H}\dot{A} && \text{(A-8)}
\end{aligned}
$$

Solving equations above, we therefore have

$$
A = 2\zeta, E = \frac{2\dot{\zeta}}{H}, F = -\frac{\zeta}{aH} + \epsilon a \nabla^{-2}\dot{\zeta} \tag{A-9}
$$

where $\epsilon \equiv -\frac{\dot{H}}{H^2}$. In other words, we can write the metric and its fluctuation in terms of ζ as

$$
\begin{aligned}
g_{00} &= -\Big(1 + \frac{2\dot{\zeta}}{H}\Big) = N_i N^i - N^2 && \text{(A-10)}\\
g_{0i} &= \partial_i\Big(-\frac{\zeta}{H} + \epsilon a^2 \nabla^{-2}\dot{\zeta}\Big) = N_i && \text{(A-11)}\\
g_{ij} &= a^2\delta_{ij}\Big(1 + 2\zeta\Big) = h_{ij} && \text{(A-12)}
\end{aligned}
$$

The determinant of the metric is

$$
\sqrt{-g} = N\sqrt{h} = a^3(1 + \frac{\dot{\zeta}}{H})e^{3\zeta} \tag{A-13}
$$

[2] Pg. 8.1 – 9 of [9]

The gravitational, inflaton, and matters actions in $\delta\varphi = 0$ gauge are

$$\mathcal{L} = \frac{\sqrt{-g}}{2}\Big[\dot{\bar{\varphi}}^2 + 2V(\bar{\varphi}) + \frac{R}{8\pi G}\Big] + \mathcal{L}_M(\sigma, \chi, \bar{\psi}\psi, A_\mu A^\mu) \tag{A-14}$$

where $\mathcal{L}_M(\sigma, \chi, \bar{\psi}\psi, A_\mu A^\mu)$ are the additional matters such as minimal coupled scalar, conformal scalar, fermion, and gauge fields that do not have the background. The first three terms give vertices of purely gravity (or purely ζ in the gauge which inflaton does not fluctuate $\delta\varphi = 0$). We are presently interested in the interactions of matter and gravitational fluctuations in the last term ($\mathcal{L}_M$ term) because in general the matter loops are larger than the gravity loops by a factor of $8\pi G$. Therefore, the time dependent tri-linear vertices of general matter is

$$\begin{aligned} H_{\zeta MM}(t) &= -\frac{1}{2}\int d^3x a^3 T^{\mu\nu}\delta g_{\mu\nu} && \text{(A-15)} \\ &= \int d^3x a^3\Big(\frac{\dot{\zeta}}{H}T^{00} - \partial_i(-\frac{\zeta}{H} + \epsilon a^2\nabla^{-2}\dot{\zeta})T^{0i} - a^2\zeta T^{ii}\Big) && \text{(A-16)} \end{aligned}$$

where $T^{\mu\nu}$ is the energy momentum tensor of arbitrary matter evaluated at quadratic order in fluctuations. With the Bianchi Identity,

$$T^{\mu\nu}_{;\nu} = T^{\mu\nu}_{,\mu} + \Gamma^{\mu}_{\mu\lambda}T^{\lambda\nu} + \Gamma^{\nu}_{\mu\lambda}T^{\mu\lambda} = 0 \tag{A-17}$$

we have

$$\frac{1}{a^3}\frac{d}{dt}(a^3T^{00}) + a\dot{a}T^{ii} + \partial_i T^{i0} = 0 \tag{A-18}$$

where we use $\bar{\Gamma}^i_{i0} = 3H, \bar{\Gamma}^0_{ij} = a\dot{a}\delta_{ij}, \bar{\Gamma}^0_{0\lambda} = \bar{\Gamma}^0_{i0} = \bar{\Gamma}^i_{ij} = 0$ for the unperturbed FRW metric. Integrating by part in space and using Bianchi Identity eq. (A-18), eq.

(A-15) becomes

$$H_{\zeta MM}(t) = \int d^3x a^3 \Big(\frac{\dot{\zeta}}{H} T^{00} + (\frac{\zeta}{Ha^3} - \frac{\epsilon}{a} \nabla^{-2}\dot{\zeta}) \frac{d}{dt}(a^3 T^{00}) \tag{A-19}$$
$$-\epsilon H a^2 \nabla^{-2}\dot{\zeta}(a^2 T^{ii}) \Big) \tag{A-20}$$

where the term $a^2\zeta T^{ii}$ is cancelled. With Mukhanov equation or the wave equation in interaction picture

$$\ddot{\zeta} + [3H + \frac{\dot{\epsilon}}{\epsilon}]\dot{\zeta} - \frac{\nabla^2}{a^2}\zeta = 0 \tag{A-21}$$

Eq. (A-19) is simplified as

$$H_{\zeta MM}(t) = -\epsilon H a^5 \int d^3x (T^{00} + a^2 T^{ii}) \nabla^{-2}\dot{\zeta} + \dot{Y} \tag{A-22}$$

where

$$Y(t) = a^3 \int d^3x T^{00} \Big(\frac{\zeta}{H} - \epsilon a^2 \nabla^{-2}\dot{\zeta} \Big) \tag{A-23}$$

The gravitational and general matters interactions in eq. (A-22) are what we used in chapters $3-6$ because they are more simplified than the direct expansion of fluctuations.

A.2 Adding Matters and Inflaton Interactions: More General Inflaton Potentials

In the forgoing section we have considered an inflaton potential $V(\varphi)$ and the interactions of gravity and matters only. The most general and realistic theories of quantum fluctuations are the theories that consider *all* matters interacting with gravity *and* inflaton. In this section we study what happens when an inflaton potential is more general such that

$$V(\varphi) \to V(\varphi, \sigma, \bar{\psi}\psi, A_\mu A^\mu) \tag{A-24}$$

where matters interact with both inflaton and gravitational fluctuations.
The same approach shown in the previous section can be easily extended to this more general inflaton potential without redoing the entire calculations. This is an additional advantage of choosing Maldacena gauge, a gauge which inflaton does not fluctuate ($\delta\varphi = 0$). The calculation for this gauge is less complicated than that of other gauges. We do not have to worry about the additional interactions between $\delta\varphi$ and other matters fluctuations.

$\star$ Even when the additional interactions of inflaton and matters arise, the results in eqs. (8.2), (8.6), and (8.11) do not change but only the masses are shifted by

$$m_\psi \quad \to \quad m_\psi + \frac{\partial^2 V}{\partial\bar{\psi}\partial\psi}|_{\psi=0} \tag{A-25}$$

$$m^2_{A_\mu} \quad \to \quad m^2_{A_\mu} + \frac{\partial^2 V}{\partial A_\mu \partial A^\mu}|_{A_\mu=0} \tag{A-26}$$

$$m^2_\sigma \quad \to \quad m^2_\sigma + \frac{\partial^2 V}{\partial\sigma^2}|_{\sigma=0} \tag{A-27}$$

The is because we have chosen a gauge which inflaton does not fluctuate $\delta\varphi = 0$ and the mass shift($\frac{\partial^2 V}{\partial\bar{\psi}\partial\psi}|_{\psi=0}$, $\frac{\partial^2 V}{\partial A_\mu \partial A^\mu}|_{A_\mu=0}$ or, $\frac{\partial^2 V}{\partial\sigma^2}|_{\sigma=0}$), which is a function of unperturbed inflaton only, does not change much during inflation. We therefore can approximate the unperturbed inflaton at the time of horizon exit $\bar{\varphi}(t) \simeq \bar{\varphi}(t_q)$. Hence there is no additional consequence to the momentum dependence of loop spectrums.

Therefore, the spectrum is nearly scale invariant even if we add interactions of arbitrary matters and inflaton to the interactions of matters and gravity.

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Vita

Kanokkuan Chaicherdsakul, the youngest child of Theinchai and Nantana, was born in October 3, 1978 in Bangkok Thailand. She graduated with a B.S.(first class honors) degree in physics from Mahidol University and was awarded a six-year scholarship from the Royal Thai Government agency to pursue a PhD degree in the USA. She began her graduate studies in physics at The University of Texas at Austin at the age of 20 in August 1999. During the summers of 2001, 2004, and 2005, she attended the TASI, SLAC(Stanford University), and YKIS schools/conferences respectively. For the early part of her graduate studies, she worked as a Teaching Assistant, assisting professors on grading homework and exams. She worked with Prof. Bryce DeWitt on her qualifying research. Her previous publications were in hep-th /0207240 and American Journal Physics, Sep. 2001, Vol **69**, Issue 9, pp 996 – 1009. For the later part of her graduate studies, she has been working as a graduate research assistant in the Theory Group. Her recent research is on quantum cosmology in inflationary theories with Prof. Steven Weinberg. Her recent work will be published in Physical Review D.

Permanent Address: 10 Soi.Charernnakorn 49 Bangkok Thailand 10600

This dissertation was typeset with LaTeX 2_ε[3] by the author.

[3]LaTeX 2_ε is an extension of LaTeX. LaTeX is a collection of macros for TeX. TeX is a trademark of the American Mathematical Society. The macros used in formatting this dissertation were written

by Dinesh Das, Department of Computer Sciences, The University of Texas at Austin, and extended by Bert Kay, James A. Bednar, and Ayman El-Khashab.